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n-Dimensional Fuzzy Implications: Analytical, Algebraic and Applicational Approaches

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n-Dimensional Fuzzy Implications:
Analytical, Algebraic and Applicational Approaches

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Start by doing what is necessary, then what is possible, and suddenly you are doing the impossible.
— SAINT FRANCIS OF ASSISI

ABSTRACT

ZANOTELLI, Rosana Medina. **n-Dimensional Fuzzy Implications: Analytical, Algebraic and Applicational Approaches**. Advisor: Renata Hax Sander Reiser. 2020. 117 f. Thesis (Doctorate in Computer Science) – Technology Development Center, Federal University of Pelotas, Pelotas, 2020.

The n-dimensional fuzzy logic (n-DFL) is an extension of the fuzzy logic (FL) as old as hesitant fuzzy sets and less exploited, motivating new investigations, and promoting results to consolidate this research area. The study of n-DFL contributes to overcome the insufficiency of traditional fuzzy logic in modeling imperfect and imprecise information coming from different opinions of experts. Moreover, the possibility to model repeated and ordered degrees of membership in the n-dimensional fuzzy sets is considered a consolidated strategy in applied technologies including areas as pattern recognition, image processing, data mining and mathematical morphology. This large field of applications motivate the studies developed in this work. Based on representability of n-dimensional fuzzy connectives, we are able to extend relevant theoretical results from fuzzy connectives to n-dimensional fuzzy approach. In particular, this proposal introduces the study of n-dimensional fuzzy implications (n-DI) following distinct approaches: (i) analytical studies, defining n-DI, presenting the most desirable properties as neutrality, ordering, (contra-)symmetry, exchange and identity principles, and also discussing their interrelationships and exemplifications; (ii) algebraic aspects mainly related to left- and right-continuity of representable n-dimensional fuzzy t-norms and the generation of n-DI from existing fuzzy implications; (iii) n-dimensional approach of fuzzy implication classes explicitly represented by fuzzy connectives, as (S,N)-implications and QL-implications; (iv) prospective studies of n-dimensional R-implications (n-DRI), analyzing extended conditions to verify the residuation principle and their characterization based on n-dimensional t-norms and n-DI; (v) constructive method obtaining n-DRI based on n-dimensional aggregation operators, presenting an exemplification in the solution of a problem involving CIM-MCDM, based on Łukasiewicz n-DRI; and also including (vi) an introductory study considering an n-DI in modeling approximate reasoning of inference schemes, dealing with the effecting role of such connectives in based-rule fuzzy systems. In addition, taking into account the action of automorphism and fuzzy negations on $L_n(U)$, conjugate and dual operators of n-DI can be respectively obtained, preserving algebraic and analytical properties.

Keywords: n-dimensional fuzzy sets. n-dimensional fuzzy implications. n-dimensional intervals. n-dimensional (S,N)-implications. n-dimensional QL-implications. n-dimensional R-implications. approximate reasoning.

RESUMO

ZANOTELLI, Rosana Medina. **Implicações Fuzzy n-Dimensionais: Abordagens Analítica, Algébrica e Aplicacional**. Orientador: Renata Hax Sander Reiser. 2020. 117 f. Tese (Doutorado em Ciência da Computação) – Centro de Desenvolvimento Tecnológico, Universidade Federal de Pelotas, Pelotas, 2020.

A lógica fuzzy n-dimensional (n-DFL) é uma extensão da lógica fuzzy (FL) tão antiga quanto conjuntos fuzzy hesitantes e menos explorada, motivando novas investigações e promovendo resultados para consolidar essa área de pesquisa. O estudo do n-DFL contribui para superar a insuficiência da lógica fuzzy tradicional na modelagem de informações imperfeitas e imprecisas provenientes de diferentes opiniões de especialistas. Além disso, a possibilidade de modelar graus de pertinência repetidos e ordenados nos conjuntos fuzzy n-dimensionais é considerada uma estratégia consolidada em tecnologias aplicadas, incluindo áreas como reconhecimento de padrões, processamento de imagens, mineração de dados e morfologia matemática. Esse amplo campo de aplicações motiva os estudos desenvolvidos neste trabalho. Com base na representabilidade dos conectivos fuzzy n-dimensionais, somos capazes de estender os relevantes resultados teóricos dos conectivos fuzzy à abordagem fuzzy n-dimensional. Em particular, esta proposta introduz o estudo das implicações fuzzy n-dimensionais (n-DI), seguindo abordagens distintas: (i) estudos analíticos, definindo n-DI, apresentando as propriedades mais desejáveis como neutralidade, ordenação, (contra)-simetria, princípios de troca e identidade e também discutindo suas inter-relações e exemplificações; (ii) aspectos algébrico relacionados principalmente à continuidade esquerda e direita de t-normas fuzzy n-dimensional representável e a geração de n-DI a partir de implicações fuzzy existentes; (iii) abordagem n-dimensional de classes de implicações fuzzy representadas explicitamente por conectivos fuzzy, como (S,N)-implicações e QL-implicações; (iv) estudos prospectivos de R-implicações n-dimensionais (n-DRI), analisando condições estendidas para verificar o princípio da residuação e sua caracterização com base nas t-normas n-dimensionais e n-DI; (v) método construtivo de obtenção de n-DRI com base em operadores de agregação n-dimensionais, apresentando uma exemplificação na solução de um problema envolvendo CIM-MCDM, com base em n-DRI Łukasiewicz; e também incluindo (vi) um estudo introdutório considerando um n-DI na modelagem de esquemas de inferência do raciocínio aproximado, lidando com o papel efetivo de tais conectivos em sistemas fuzzy baseado em regras. Além disso, considera a ação do automorfismo e das negações fuzzy em $L_n(U)$, operadores duais e conjugados de n-DI podem ser obtidos respectivamente, preservando as propriedades algébricas e analíticas.

Palavras-chave: conjuntos fuzzy n-dimensionais. implicações fuzzy n-dimensionais. intervalos n-dimensionais. (S,N)-implicações n-dimensionais. QL-implicações n-dimensionais. R-implicações n-dimensionais. raciocínio aproximado.

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LIST OF ABBREVIATIONS AND ACRONYMS

AO	Admissible Order
AR	Approximate Reasoning
AFS	Atanassov's Intuitionistic Fuzzy Set
CIM	Computer-Integrated Manufacturing
CL	Complete Lattice
CRI	Compositional Rule of Inference
DM	Decision Making
DMP	Decision Making Problem
DCPO	Distributive Complete Partial-Ordered Lattice
MFFMCC	Formal Methods and Mathematical Fundamentals of Computer Science
I	Fuzzy Implication
FL	Fuzzy Logic
FN	Fuzzy Negation
FS	Fuzzy Set
FST	Fuzzy Set Theory
GDMP	Group Decision Making Problems
HES	Hesitation
HFS	Hesitant Fuzzy Set
IMP	Imperfect
IND	Indeterminacy
IVAFS	Interval-Valued Atanassov Intuitionistic Fuzzy Set
IVFS	Interval-valued Fuzzy Set
IFIx	Intuitionistic Fuzzy Index
IFN	Intuitionistic Fuzzy Negation
LC	Left-continuit
LEM	Law of Excluded Middle
LoLiTA	Logic, Language, Information, Theory and Applications
LUPS	Laboratory of Ubiquitous and Parallel Systems

MAGDMP	Multiple Attribute Group Decision Making Problems
MCDM	Multiple Criteria and Decision Making Problems
MEDM	Multi-Expert Decision Making
MISO	Multi-Input Single-Output
N	Negação
n-DA	n-dimensional Aggregation
n-DFJ	n-dimensional Fuzzy Coimplicator
n-DFL	n-dimensional Fuzzy Logic
n-DFPR	n-dimensional Fuzzy Preference Relations
n-DFS	n-dimensional Fuzzy Set
n-DFSI	n-dimensional Fuzzy S-implication
nDFSr	n-dimensional Fuzzy Set without repetition
n-DI	n-dimensional Fuzzy Implication
n-DIS	n-dimensional Interval Sequence
n-DN	n-dimensional Fuzzy Negation
n-DRI	n-dimensional R-implication
n-DS	n-dimensional Triangular Conorms
n-DT	n-dimensional Triangular Norms
RP	Residuation Principle
S	Triangular Conorms
SFN	Strong Fuzzy Negation
SISO	Single-Input Single-Output
Sn-DN	Strong n-dimensional Fuzzy Negation
T	Triangular Norms
T1FS	Type-1 Fuzzy Set
T2FL	Type-2 Fuzzy Logic
T2FS	Type-2 Fuzzy Set
THFS	Typical Hesitant Fuzzy Set
TnFS	Type-n Fuzzy Set
TR	Theoretical Research
UNC	Uncertainty
UNCn	n-Component Uncertainty
VS	Vague Set

LIST OF SYMBOLS

M	Aggregation function
T_P	Algebraic product
x	Any element
ψ	Automorphism on U
φ	Automorphism on $L_n(U)$
$\mathcal{C}_{I_{S,N}}$	Class of all S-implications
f	Function
g	Function
A	Fuzzy aggregate function
I_N or J	Fuzzy coimplication
π	Fuzzy index
I_{KD}	Kleene Dienes implication
I_{LK}	Łukasiewicz implication
S_M	Maximum T-conorm
μ	Membership function
T_M	Minimum T-norm
f_N	N-dual function f
S_P	Probabilistic sum
$I_{T,S,N}$	QL-implication
I_{RC}	Reichenbach implication
$L_n(U)$	Set n-dimensional intervals
$I(U)$	Set of all fuzzy implications
$\mathcal{I}(L_n(U))$	Set of all n-dimensional fuzzy implications
U	Set unit interval [0,1]
$I_{S,N}$	S-Implication or strong implication
N_S	Standard negation or Zadeh negation
S_{LK}	T-conorm Łukasiewicz
T_N	T-conorm N dual of T

S_{nM}	T-conorm nilpotent maximum
T_{LK}	T-norm Łukasiewicz
S_N	T-norm N dual of S
T_{nM}	T-norm nilpotent maximum
χ	Universe set
I_{WB}	Weber implication

1 INTRODUCTION

The central idea in the seminal Zadeh's research work is that "in fuzzy logic (FL) everything is allowed to be a matter of degree", meaning that even the degree could be defined as a fuzzy set. In such extended approach, in his 1971's article (ZADEH, 1971) Zadeh conceived the main idea of type-n fuzzy sets (TnFS). In 1975, Zadeh presented the formal definition of TnFS (ZADEH, 1975) whose relevance emerges from the insufficiency of the traditional fuzzy logic (FL) in modeling inherent imperfect information in order to define antecedent and consequent of membership functions in an inference system.

Thus, this general concept includes the type-2 fuzzy sets (T2FS) whose memberships function are defined from a non-empty universe χ to the set of all fuzzy set (FS), or equivalent, of all type-1 fuzzy set (T1FS). Since then, several different extensions from FS to T2FS are considered. Following the structure of type-2 fuzzy logic (T2FL) systems as proposed by (KARNIK; MENDEL, 1998), the logical approach of n-dimensional fuzzy sets is proposed in order to investigate corresponding extension of logical connectives.

According with (SHANG; YUAN; LEE, 2010), the theory of n-dimensional fuzzy sets (n-DFS) is a particular case of the theory of fuzzy multisets (YAGER, 1986), related to a fixed number n of membership values, which are assigned to each element of a non-empty universe and ordered in an increasing way (BEDREGAL et al., 2012).

In the case of n-DFS, the different membership values are considered as an n-tuple named n-dimensional interval providing an interpretation for n-levels of uncertainty in corresponding memberships degrees (MEZZOMO; MILFONT; BEDREGAL, 2018). It is also a particular case of the concept of multidimensional fuzzy intervals, a new fuzzy set extension in which its elements (intervals) can have any dimensions (LIMA, 2019).

Taking into account that in a decision-making problem group we have as many evaluations as decision makers, n-dimensional fuzzy sets are suitable models for representation of uncertainty (imprecision, inexactness, ambiguity) intrinsic to multiple attributes, modelling possible repetitive opinions of many experts in decision making problems (DMP). However, in the interpretation related to an n-dimensional interval, each membership function has a fixed dimension, restricting the modelling in which its elements (intervals) have a fixed n-dimension.

By recognizing the value of studying fuzzy connectives in logical structuring of fuzzy systems, mainly based on fuzzy negations (FODOR; ROUBENS, 1994), triangular norms and conorms (E. TRILLAS C. ALSINA, 2005; KLEMENT; MESIAR; PAP, 2004), implications and coimplications (BACZYŃSKI; JAYARAM, 2008; BACZYŃSKI; JAYARAM, 2009), bi-implications and bi-coimplications (TRILLAS; VALVERDE, 1981), difference and co-difference (ZANOTELLI et al., 2018; HUAWEN, -; ROBERTS, 1987), aggregation operators (DETYNIECKI, 2001) and so many other fuzzy operators, this work aims to extend these studies in the context of n-dimensional interval approach.

In particular, fuzzy implications are one of the main operations in multi-valued fuzzy logic, generalizing the classical notion of implications, which can be applied in approximate reasoning (AR) models and inference schemes of deductive systems, decision support systems and formal methods of proof systems (YAGER, 2004a).

In order to contribute with the theoretical research on n-dimensional intervals, this thesis considers the study of n-dimensional fuzzy implications not only as an extension of fuzzy implications, but also investigating their properties, exploring main classes and also reporting application approaches in decision making systems (DMS).

1.1 Objectives

This section presents the relevant ideas of this doctoral thesis, introducing the study of n-dimensional fuzzy implications, considering the following approaches.

1. The analytical study, defining n-dimensional fuzzy implications (n-DI), presenting the most desirable properties as neutrality, ordering, (contra)symmetry, exchange and identity principles, and also discussing their interrelationships and exemplifications;

2. The algebraic aspects mainly related to left- and right-continuity of representable n-dimensional fuzzy t-norms and the generation of n-dimensional fuzzy implications from existing fuzzy implications;

3. The n-dimensional approach of fuzzy implication classes explicitly represented by fuzzy connectives, as (S, N)-implications and QL-implications;

4. The prospective study of n-dimensional R-implications, analysing extended conditions to verify the Residuation Principle and the action of n-dimensional automorphism generating a new n-DRI and characterizing n-DRI based on n-dimensional t-norms and n-dimensional fuzzy implications is discussed;

5. The constructive method obtaining n-DRI based on n-dimensional aggregation operators, including and exemplification in the solution to a problem involving CIM-MCDM, which is based on the residual Łukasiewicz n-dimensional fuzzy implication;

6. The study of n-DI in approximate reasoning of inference schemes, dealing with the effecting role of n-DRI in based-rule fuzzy systems.

Thus, this thesis aims to contributes to the study of n-dimensional fuzzy implications, introducing related axiomatic definitions, extending main properties, analysing their representability from fuzzy implications and also considering the study of main classes which are obtained by action of aggregations and negation operators. The classes of explicit/implicit representation as (S, N)-implication, QL-implication joint with R-implication are extended from their n-dimensional fuzzy approaches.

The studied properties are mainly related to algebraic and mathematical analysis approaches. The former is largely (but not restricted) a broad area, combining fuzzy sets joint with operations and their properties. The latter focuses more on studying the continuity, closed related to metric and distances, supporting measures to perform comparison, correlation, ordering, etc. on multi-valued fuzzy sets.

1.1.1 Methodology

This work proposes an incremental methodology based on theoretical studies structuring the main propositions/theorems of the thesis.

It aims to establish the extension of main classes of n-dimensional fuzzy implications, consolidating this approach of fuzzy logic, treating different levels of uncertainty in degrees of membership functions in fuzzy applications, thus modeling the imperfect information or giving more credibility to the information of a system.

The activities related for the proposal methodology are briefly described in the following:

1. Consolidation of the research of n-dimensional fuzzy implications;

2. Extension of main properties of fuzzy implications based on the notion of representability of n-dimensional fuzzy implications;

3. Extension of main classes of fuzzy implications on $L_n(U)$;

4. Exploration of conjugate and dual operators in order to obtain new expressions of n-dimensional fuzzy implications;

5. Introduction of the extension from U to $L_n(U)$ for the AR based on expression of n-dimensional fuzzy implications.

Additionally, this thesis promotes presentation and publication of new results in conferences, congresses and journals of related research area, expanding contacts and cooperation of LUPS, MFFMCC research group, strengthening relations with LoLiTA (UFRN/Brazil) and GIARA (Navarra/Spain) research groups.

1.2 Motivations

The use of fuzzy implication classes plays an important role in FL, since they are applied in fuzzy control, imprecision analysis from natural language and soft-computing techniques. Such use can be extended to n-dimensional interval-valued fuzzy logic.

In addition, fuzzy implications provide the mathematical basis for the generation of powerful techniques for AR involving uncertain, vague and nebulous processes. Moreover, the mathematical formalism of fuzzy implications and other connectives of extensions of fuzzy logic together with the ability to model inaccurate information related to human reasoning assist in decision making (DM) based on multi-attributes in control systems, often related to information obtained by multiple specialists (XIA; XU; CHEN, 2013).

This large field of applications motivates the extended studies developed in this work, contributing as new theoretical results to underlying applications in areas as pattern recognition (BISHOP, 2006), image processing (CELENK, 1990), data mining (KUMAR STEINBACH M, 2006) and mathematical morphology (DUBOIS; OSTASIEWICZ; PRADE, 2000; KLIR, 2005). As another relevant motivation, we consider to study the extension of fuzzy implications to the n-dimensional interval fuzzy approaches, consolidating projects connected to Laboratory of Ubiquitous and Parallel Systems (LUPS), Formal Methods and Mathematical Fundamentals of Computer Science (MFFMCC), LoLiTA (UFRN/Brazil) and GIARA (Navarra/Spain), research groups in multi-valued fuzzy logics showing the conditions under which the main properties of the former sets are preserved when worked with the latter ones.

As n-dimensional fuzzy intervals are an extension of fuzzy logic as old as hesitant fuzzy sets (HFS) but less exploited, new investigations will promote results and contribute to consolidate this area. Additionally, the study of n-dimensional fuzzy logic (n-DFL) contributes to overcome the insufficiency of traditional fuzzy logic in modeling imperfect information coming from different opinions of experts. Moreover, the possibility to model repeated and ordered degrees of membership in the n-dimensional fuzzy sets (n-DFS) is considered a wide strategy in applied technologies.

By making use of representability of n-dimensional fuzzy connectives, we are able to extend relevant theoretical results from fuzzy connectives to n-dimensional fuzzy approach.

1.2.1 Research proposal

This thesis discusses the problem of extending fuzzy implications from lattice of fuzzy values $(U, \leq)$ to the n-dimensional upper simplex $(L_n(U), \leq)$ also providing a suitable extension and enabling to preserve the largest number of their properties. The extension of fuzzy functions or operators arises in many situations in various branches of computer science. See, e.g. the applied research in relevant areas:

(i) image processing - considering classes of fuzzy operators which aim at extending the range use of fuzzy techniques in image processing applications;

(ii) computational semantics - making use of fuzzy propositional computation to model some uncertain and ambiguous in semantic problems; and

(iii) mathematical morphology - modeling erosion and dilation operators.

Moreover, multi-valued fuzzy logic have been explored in the applied area, providing an adequate logical structure to solve multiple criteria and decision making problems (MCDM), dealing with uncertainty (imprecision, inexactness, ambiguity, hesitation) which are intrinsically connected to multiple attributes and many experts.

The relevance of this proposal can be credited to the theoretical research underlying this applied point of view, extending fuzzy connectives to their n-dimensional interval approach. In particular, considering the study of n-dimensional fuzzy implications and their main properties.

In this context, the integration of new results consolidating this new research area in n-dimensional fuzzy sets are mainly related to explicit and implicit classes of n-dimensional fuzzy implications.

Related operators obtained by dual and conjugate constructors on other n-dimensional fuzzy connectives as t-norms, t-conorms and fuzzy negations are also considered.

The proposal introduces the n-dimensional extension of (S, N)-implications, QL-implications and R-implications, discussing main properties characterizing such classes.

1.3 Outline of the doctoral thesis

This thesis is organized in eight Chapters, starting from preliminary aspects of fuzzy logic and evolving to more specific ones in order to develop the research.

In Chapter 2, the contributions of the type-n fuzzy sets to the systems development area are treated, a comparative and hierarchical analysis of the fuzzy sets is made showing the different interpretations of information and a literature review reporting relevant studies on the n-dimensional intervals.

The fundamental concepts of fuzzy logic is shown in Chapter 3, starting with automorphisms, connectives in fuzzy logic such as negations, triangular norms and conorms and implications with the main classes.

Fundamental concepts of n-dimensional fuzzy logic, such as order relations, the main properties of continuity of functions, automorphism, and an extension of the fuzzy sets are reported in Chapter 4.

In Chapter 5, the n-dimensional fuzzy implications are reported, showing the main properties as well as representability and finally the expansion of the classes (S, N)-implication and QL-implication.

Focusing on the R-implications, the discussions extending the residuation property based on Moore-metric, continuity and monotonicity properties with respected to the usual partial orders on $L_n(U)$ are considered in Chapter 6. Extending results from interval-valued fuzzy set theory to n-dimensional simplex, the class of n-DRI is constructed based on n-DA aggregation operators, given as the minimum operator and left-continuous t-norms. Some topics refer to the residuation property, conjugate operators, characterization of n-DRI, application of of n-DA to obtain n-DRI and main properties preserved by such methodology.

In Chapter 7, some basic concepts of approximate reasoning are extended to the n-dimensional fuzzy approach, showing that n-dimensional fuzzy deduction rules generalize fuzzy rules of classical logic.

In Chapter 8, final considerations are presented, highlighting the originality and relevance of the thesis, the main contributions, the publications made during the doctorate course and the possibilities for future work to be developed.

2 n-DIMENSIONAL INTERVALS: CONTEXT AND RELEVANT CONTRIBUTIONS

The central idea in Zadeh's research work is that "in FL everything is allowed to be a matter of degree" meaning that even the degree could be defined as a fuzzy set (FS) (BUSTINCE et al., 2016). In his 1971's article (ZADEH, 1971) Zadeh conceived the main idea of the type-n fuzzy sets (TnFS), for $n \in \mathbb{N}$, meaning that a membership degree of a TnFS can be expressed as a FS or equivalent, T1FS.

In (BUSTINCE et al., 2016) a role of 22 extensions of FS (ZADEH, 1973) is presented. Selected from that, we briefly describe the main FS considered in this thesis.

Zadeh (ZADEH, 1975) formalizes the theoretical concepts of the T2FL, whose relevance emerges from the insufficiency of the traditional fuzzy logic in modeling inherent imperfect information in order to define antecedent and consequent of membership functions in T1FS inference system.

2.1 Contributions on type-n fuzzy sets

As a general concept, the theoretical approach of TnFS includes the theory Type-2 Fuzzy Sets (T2FS) whose memberships function are defined from a non-empty universe χ to the set of all FS, or equivalent, of all T1FS. Since then, several different extensions from FS to T2FS are considered.

As a special class of extended fuzzy sets introduced by Zadeh, T2FSs were mathematically defined by Mendel and Karnik in 1998 (KARNIK; MENDEL, 1998) including the study of first operations on such sets. So, T2FL are fuzzy systems in which at least one of their antecedent or consequent are T2FS (KARNIK; MENDEL, 2001).

Thus, in this new and comprehensive approach, the uncertainty about the membership function is defined as T1FS. Such approach contributes to the generalization of the fuzzy set theory, since if there is no uncertainty in the membership function, then a T2FS is reduced to a T1FS (DUBOIS; PRADE, 2005).

TnFS collaborate more significantly in the modeling of fuzzy systems involving the random data approximation in temporal evolution (MENDEL, 2017). Moreover, relevant applications based on T2FL (DESCHRIJVER; KERRE, 2005; ARIELI et al., 2003) contribute to identify models or prediction of behaviour from expert information. Such logical approach has also been applied in the following technologies:

- controlling a mobile robot (SANCHEZ; CASTILLO; CASTRO, 2015), (BUSTINCE et al., 2016) ;
- providing prediction of the survival time of myeloma patients based on hybrid systems using T2FL and optimization with genetic algorithms (QIU, 2006);
- modelling expert system for the realization of shopping via web (GU; ZHANG, 2007);
- developing specialized systems for analysis and estimation of the survival time of wireless sensor networks (SHU; LIANG; GAO, 2008).

2.1.1 Comparative and hierarchical analysis in TnFS

See in Table 1 a summary of distinct levels of TnFS and corresponding interpretation of imperfect information in modeling inference systems.

In 1983, the Atanassov's intuitionistic fuzzy sets (ATANASSOV, 1986) (AFS) were defined as a special type of T2FSs, in such a way that each element is associated with a pair of real numbers, representing the membership and non-membership degrees, which is not necessarily defined by the complementary relationship.

Based on Atanassov's (ATANASSOV; GARGOV, 1998) approach, each element of an AFS is associated with its corresponding intuitionistic fuzzy index (IFIx) (ATANASSOV, 1989), indicating not only its uncertainty but also the hesitance in the corresponding AFS, which is related to its membership and non-membership degrees, respectively.

Basic ideas of interval-valued fuzzy sets (IVFS) were simultaneously defined by Zadeh (ZADEH, 1975) and Sambuc (SAMBUC, 1975), proposing the concept of ϕ-Flou sets as an analogous notion of IVFS. Moreover, many other relevant works in IVFS were preliminarily studied by Mizumoto and Tanaka (MIZUMOTO; TANAKA, 1976) also considering Dubois and Prade's (DUBOIS; PRADE, 2005) and (DESCHRIJVER; KERRE, 2005) research.

In 1989 Atanassov and Gargov presented the notion of interval-valued Atanassov intuitionistic fuzzy sets (IVAFS) (ATANASSOV; GARGOV, 1998), which extends AFS in the field of T2FSs. Since then, applications of AFS and IVAFS have been developed (ZHANG; XU, 2012), modeling problems of DM (ZHANG, 2013) based on multi-criteria (CHEN, 2012) and related to many areas. Moreover, cognitive psychology and medicine, for instance, also including image processing for exams and diagnosis of patients.

Table 1 – Interpretation of information

TnFS	Fuzzy Representation of Imperfect Information					Fuzzy Values
	UNC	IND	IMP	HES	UNC$_n$	
FS (ZADEH, 1965)	X					U
AFS (ATANASSOV, 1986)	X	X				$\tilde{U}$
IVFS (ZADEH, 1975)	X		X			$\mathbb{U}$
IVAFS (ATANASSOV, 1989)	X	X	X			$\tilde{\mathbb{U}}$
$\overline{nDFS_r}$ (SHANG; YUAN; LEE, 2010)	X	X			X	L_n
HFS (TORRA, 2010)	X			X		$\mathbb{H}$

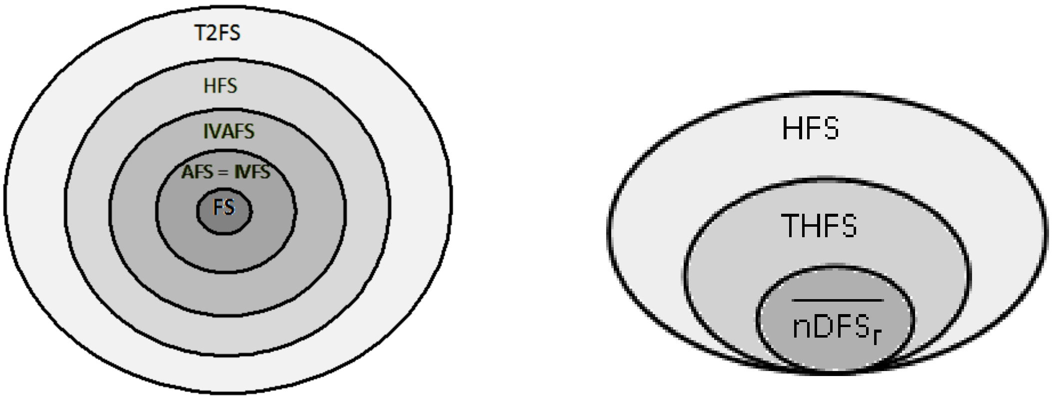
Vicenç Torra proposes hesitant fuzzy sets (HFS) as an extension of the fuzzy sets in (TORRA; NARUKAWA, 2009; TORRA, 2010) with the goal of considering fuzzy sets where the membership degree can be expressed as a set of values, enabling the description of distinct opinions of several experts. Its main motivation is concerned with the situations in which the experts provide different membership degrees and it is not always possible to obtain a consensus to unify or even aggregate these values (XIA; XU, 2011; XIA; XU; CHEN, 2013). Later, Bedregal et al. (BEDREGAL et al., 2014) introduced the notion of typical hesitant fuzzy sets (THFS) by considering only finite and nonempty hesitant fuzzy values.

In 2010, n-dimensional fuzzy sets (n-DFS) were conceived by (SHANG; YUAN; LEE, 2010) as an extension of fuzzy sets, including IVFS and IVAFS. n-DFS considers repetition of elements on the membership degrees. When the repetition of elements on the membership degrees is not considered, they can be defined as HFS. Based on (BEDREGAL et al., 2011), the main idea of an n-dimensional fuzzy set is to consider several uncertainty levels in the memberships degrees.

Additionally, the next diagram presents the inclusion structure of extension of FS considered in this work. See Figure 1(a) presenting a diagram showing the range of FS, representing inclusion relationships among different types of FS, which do not admit repetition. Moreover, Figure 1(b), shows the inclusion with respect to HFS, THFS and the complementary set of n-dimensional intervals without repetition ($\overline{nDFS_r}$) in their membership functions.

2.2 State-of-the-art on n-dimensional intervals

A literature review related to relevant studies on n-dimensional intervals was carried out in this section, taking into account keywords such as: n-dimensional fuzzy set, n-dimensional t-norms and t-conorms, n-dimensional uninorms, n-dimensional fuzzy implication, n-dimensional (S, N)-implication, n-dimensional QL-implications, n-dimensional R-implications, representability, n-dimensional automorphism, continuity



(a) FS which do not admit repetition. (b) Relation: $\overline{nDFS_r} \subseteq$ THFS $\subseteq$ HFS .

Figure 1 – Some inclusion relations in TnFS

and monotonicity properties and multidimensional fuzzy sets.

2.2.1 Related works

Significant articles were selected, highlighting their main contributions and reporting aspects related to the fuzzy reasoning adopted in diverse applied areas.

This selection also includes some previous partial results obtained in the development of this research work.

2.2.1.1 Extending the study of fuzzy implications from U to $L_n(U)$

In (ZANOTELLI; REISER; BEDREGAL, 2019/08) R-implications are studied on $L_n(U)$ considering the conditions under which main properties are preserved, and their representability from U to $L_n(U)$. Some results in the class of n-dimensional R-implications obtained from t-representable norms on $L_n(U)$ are discussed.

In (MEZZOMO et al., 2019), n-dimensional fuzzy sets are preseted as an extension of fuzzy sets where the membership values are n-tuples of real numbers in the unit interval U ordered in increasing order. The set of n-dimensional intervals is denoted by $L_n(U)$. The paper considers the notion of uninorms and n-dimensional fuzzy sets to define n-dimensional interval uninorms and we obtain results involving the notion of the neutral element, degenerate element, representable uninorms, $\subseteq$-monotone and monotone by parts. Results involving the concepts of n-dimensional interval uninorms and n-dimensional automorphisms are proven.

The main result in (ZANOTELLI; REISER; BEDREGAL, 2018a) has considered the study of n-dimensional fuzzy implications, formalizing constructions and dealing with duality and conjugation operators. Additionally, the conditions under which main properties are preserved based on their representability from U to $L_n(U)$ are presented. Some results in the class of S-implications obtained from t-representable conorms and involutive n-dimensional fuzzy negations on $L_n(U)$ are also discussed.

The results in (ZANOTELLI; REISER; BEDREGAL, 2018b) are mainly connected with main properties of n-dimensional fuzzy implications, considering some properties and exemplification. In addition n-dimensional QL-implicators are studied considering duality and conjugation operators.

2.2.1.2 Multidimensional fuzzy sets

(LIMA, 2019) has considered the study of multidimensional fuzzy sets as a generalization of fuzzy n-dimensional sets in which the elements can have different dimensions. The work presents a way to generate comparisons (orders) of elements of different dimensions, discussing conditions under which these sets have lattice structure and introduces the concepts of admissible orders, aggregation functions and multidimensional negations about multidimensional fuzzy sets.

2.2.1.3 Moore continuous metric and n-dimensional fuzzy negations

In (MEZZOMO; BEDREGAL; MILFONT, 2018), the notion of representable n-dimensional interval fuzzy negations considers the Moore-continuous approach. Additional studies analyse main conditions guarantying the existence of equilibrium point in classes of representable (Moore-continuous) n-dimensional interval fuzzy negations. Results showing that the changing of the dimensions of representable Moore-continuous n-dimensional fuzzy negations preserves their equilibrium points.

The study presented in (MEZZOMO; MILFONT; BEDREGAL, 2018) considers the definitions of (continuous) Moore-metric and n-dimensional interval fuzzy sets characterizing the notion of (continuous) n-dimensional interval Moore-metric, proving some results about them. In addition, considering the intuitive notion of strict n-dimensional interval fuzzy negation, the notion of continuity for changing of the dimensions of Moore-continuous n-dimensional interval fuzzy negations are also investigated.

2.2.1.4 n-Dimensional fuzzy negations

In (BEDREGAL; MEZZOMO; REISER, 2018), a special extension from U to $L_n(U)$, related to n-representable fuzzy negations, is studied, summarizing the class of such functions which are continuous and monotone by part. The main properties of (strong) fuzzy negations on U are preserved by representable (strong) fuzzy negation on $L_n(U)$, mainly related to the analysis of degenerate elements and equilibrium points. The conjugate obtained by the action of an n-dimensional automorphism on an n-dimensional fuzzy negation provides a method to obtain other n-dimensional fuzzy negation, in which properties as representability, continuity and monotonicity on $L_n(U)$ are preserved.

The research in (MEZZOMO et al., 2016) investigates the class of functions on

$L_n(U)$ which are continuous and strictly decreasing, called n-dimensional strict fuzzy negations. In particular, investigate the class of representable n-dimensional strict fuzzy negations, i.e., n-dimensional strict fuzzy negation which are determined by strict fuzzy negation. The main properties of strict fuzzy negations on U are preserved by representable strict fuzzy negations on $L_n(U)$. In addition, the conjugate obtained by action of an n-dimensional automorphism on an n-dimensional strict fuzzy negation provides a method to obtain other n-dimensional strict fuzzy negations, preserving properties of the original one, as well as the Fodor's characterization theorem.

2.2.1.5 n-Dimensional triangular norms and triangular conorms

The article (MEZZOMO; BEDREGAL; REISER, 2017) reports main definitions and results obtained for n-dimensional fuzzy negations, applying these studies mainly on natural n-dimensional fuzzy negations for n-dimensional triangular norms and triangular conorms. Additionally, conjugate operators are obtained by the action of an n-dimensional automorphism on an n-dimensional natural fuzzy negation and on n-dimensional triangular norm and triangular conorm. It provides a method to obtain other n-dimensional strong fuzzy negations, in which its properties on $L_n(U)$ are preserved.

In (BEDREGAL et al., 2011), the main idea in n-dimensional fuzzy sets is to consider several uncertainty levels in the memberships degrees. Triangular norms have played an important role in fuzzy sets theory, in the narrow as in the broad sense. So it is reasonable to extend this fundamental notion for n-dimensional intervals. In interval-valued fuzzy theory, interval-valued t-norms are related with t-norms via the notion of t-representability. A characterization of t-representable interval-valued t-norms is given in terms of inclusion monotonicity. In this paper generalizes the notion of t-representability for n-dimensional t-norms and provides a characterization theorem for that class of n-dimensional t-norms.

2.2.1.6 Decision making using n-dimensional fuzzy sets

In (DE MIGUEL et al., 2017), the proposal algorithm to solve group DMP using n-dimensional fuzzy sets, namely, sets in which the membership degree of each element to the set is given as an increasing n-tuple of elements. The use of these sets has naturally led to define admissible orders for n-dimensional fuzzy sets, presenting a construction method for those orders and studying OWA operators for aggregating tuples used to represent the membership degrees of the elements. Under these conditions, the algorithm is applied to a case study, showing that the exploitation phase which appears in many DM methods can be omitted by just considering linear orders between increasing n-tuple.

In (SILVA; BEDREGAL; BUSTINCE, 2015), it is presented a way to generate a class of weighted average operator from n-dimensional overlap functions and aggregation functions. These weighted average operators are used in an algorithm of a multi-attribute group DMP based on decision matrix. From a study in (MERIGO; CASANOVAS, 2011) which resulted in 12 different rankings. The goal of introducing this new method contribute with a new tool to provide new evidences for the decision makers select good alternatives.

2.2.1.7 Fuzzy grades based on n-dimensional fuzzy sets

In (BEDREGAL et al., 2012), it is presented a generalization of Atanassov's operators to higher dimensions. To do so, the concept of fuzzy set was used, which can be seen as a special kind of fuzzy multisets, to define a generalization of Atanassov's operators for n-dimensional fuzzy values (called n-dimensional intervals). It was proven that generalized Atanassov's operators also generalize OWA operators of any dimension by allowing negative weights. The results were applied to a DMP. The notions of aggregating functions were extended, in particular t-norms, fuzzy negations and automorphism and related notions for n-dimensional framework.

2.2.1.8 Study on cut sets of n-dimensional fuzzy sets

The n-dimensional fuzzy sets is proposed in (SHANG; YUAN; LEE, 2010) as a generalization of various fuzzy set concepts such as the FS, IVFS, AFS, IVAFS and three dimensional fuzzy set. In this study, the cut sets of the n-dimensional fuzzy sets and the interval-valued level cut sets of Zadeh fuzzy sets are considered, defining $n+1$-valued fuzzy sets, as structured sets having the same properties as those presented in the fuzzy set theory. New decomposition and representation theorems were established based on these newly defined cut sets, establishing the connections among the Zadeh fuzzy set, the n-dimensional fuzzy set and the finite-valued fuzzy set.

See Table 2 summarizing the main characteristics of above listed selected research papers on $L_n(U)$, reporting techniques, main contributions and the applied approach areas.

Table 2 – Distribution of papers based on n-dimensional fuzzy sets

Techniques	Contributions	Approaches
Study on n-Dimensional R-implications (ZANOTELLI; REISER; BEDREGAL, 2019/08)	Study of the properties that characterize the class of **R-implications on $L_n(U)$**.	TR DMP
Conjuntos Fuzzy Multidimensionais (LIMA, 2019)	Presents the concept of **multidimensional fuzzy sets** as a generalization of n-DFS.	TR
n-Dimensional Interval Uninorms (MEZZOMO et al., 2019)	Presents conjugate based on **n-dimensional interval uninorms** preserving their main properties.	TR

n-Dimensional Intervals and Fuzzy S-implications (ZANOTELLI; REISER; BEDREGAL, 2018a)	Study main properties characterizing the class of **S-implications on** $L_n(U)$.	TR
Towards the study of main properties of n-Dimensional QL-implicators (ZANOTELLI; REISER; BEDREGAL, 2018b)	Introduces the concept of **n-dimensional QL-implicators** considering duality and conjugation operators.	TR
Equilibrium Point of Representable Moore Continuous n-Dimensional Interval Fuzzy Negations (MEZZOMO; BEDREGAL; MILFONT, 2018)	Studies the class of representable (Moore-continuous) **n-dimensional interval fuzzy negations**.	TR
Moore Continuous n-Dimensional Interval Fuzzy Negations (MEZZOMO; MILFONT; BEDREGAL, 2018)	Characterizes the notion of (continuous) n-dimensional interval Moore-metric and **n-dimensional interval fuzzy negations**.	TR
n-Dimensional Fuzzy Negations (BEDREGAL; MEZZOMO; REISER, 2018)	Investigates an extension of n-representable fuzzy **negations on** $L_n(U)$ which are continuous and monotone.	n-DFPR MEDM
Natural n-dimensional fuzzy negations for n-dimensional t-norms and t-conorms (MEZZOMO; BEDREGAL; REISER, 2017)	Considers n-dimensional fuzzy negations studying **n-dimensional triangular norms and triangular conorms**.	TR
An algorithm for group decision making using n-dimensional fuzzy sets, admissible orders and OWA operators (DE MIGUEL et al., 2017)	Introduces the concept of **admissible order for n-dimensional fuzzy set** as well as a construction method for these orders.	GDMP AO
On n-dimensional strict fuzzy negations (MEZZOMO et al., 2016)	Investigate the class of representable **n-dimensional strict fuzzy negations**.	TR
Weighted Average Operators Generated by n-dimensional Overlaps and an Application in Decision Making (SILVA; BEDREGAL; BUSTINCE, 2015)	Provides a way to generate a class of **weighted average operator from n-dimensional** overlap functions and aggregation functions.	MAGDMP
A class of fuzzy multisets with a fixed number of memberships (BEDREGAL et al., 2012)	Define a generalization of Atanassov's operators for n-dimensional fuzzy values (called n-dimensional intervals).	DMP
A Characterization Theorem for t-Representable n-Dimensional Triangular Norms (BEDREGAL et al., 2011)	Generalize the notion of t-representability for **n-dimensional t-norms** and provide a characterization theorem for that class of n-dimensional t-norms.	TR
The n-dimensional fuzzy sets and Zadeh fuzzy sets based on the finite valued fuzzy sets (SHANG; YUAN; LEE, 2010)	Defines cut sets on n-DFS studying the **decomposition and representation theorems**.	TR

3 FUZZY LOGIC: FUNDAMENTAL CONCEPTS

The Fuzzy Set Theory (FST) was introduced in 1965 by the mathematician Lofti Asker Zadeh (ZADEH, 1965), with the publication of the article “Fuzzy Sets”, by combining the concepts from both logical approaches, The Boolean Logic and Łukasiewicz’s Logic, characterizing the attribution of membership degrees to the elements of a FS which is mainly depended on the application and the context.

As easily observed, technological resources based on Boolean logic were not enough in order to automate activities mainly related to industrial, computational, biological or chemical uncertainty nature problems.

The main advantage associated with the use and development of systems based on FL is to obtain a mathematical modeling of subjective linguistic terms such as “about’ or "around", making it possible to produce calculations even when we need to deal with inaccurate information in programming and store such vague concepts in computers (NGUYEN; WALKER, 2005).

Between 1970 and 1980 the industrial applications of FL happened with greater importance in Europe and after 1980, Japan began its use with applications in the industry, highlighting the application of fuzzy systems in 1983, as a process for water treatment proposed by Fuji Electric Company; and later Hitachi Company, developing a metro control system based on FL, which was totally operated in 1987. However, it was around 1990 that FL aroused as a greater interest in companies from the United States (NGUYEN; WALKER, 2005).

Due to the development of numerous practical possibilities and theoretical foundation together with the great commercial success of its applications, FL is considered nowadays as fundamental technical theory for a logical approach for modeling uncertainty with wide acceptance in the area control and DM for computational processes (KLIR, 2005).

FL approach is considered as a relevant tool to support both DM and control theory research areas, promoting greater credibility and reliability to the system developed mainly related to optimization and planning in technological areas. For instance, by considering the development of electronic components, such as elevator control (Hitachi, Fujitech and Mitsubishi), signal analysis applied in TV image adjustment (Sony), camera autofocus video (Canon), video image stabilizer (Panasonic) and even fraud detection on credit cards (DRIANKOV; HELLENDOORN; REINFRANK, 2013).

Additionally, FL also provides fundamental logical systems in many other relevant research areas such as artificial intelligence, natural language, expert systems, neural networks, etc (DUBOIS; OSTASIEWICZ; PRADE, 2000).Main concepts in fuzzy logic

3.1 Main concepts in fuzzy logic

Consider an universe set $\chi \neq \emptyset$ in the following definition of a type-1 fuzzy set.

Let $U = [0, 1]$ be the set of all fuzzy values considering the usual partial $\leq$-order of real numbers, restricted to U. Additionally, the class of FS on the universe set χ, denoted as $FS(\chi)$. Moreover, according with (BUSTINCE et al., 2016), in the following proposition, consider the partial-order relation $\leq_{FS}$ on $FS(\chi)$:

$$A \leq_{FS} B \Rightarrow \mu_A(\mathrm{x}) \leq \mu_B(\mathrm{x}), \forall \mathrm{x} \in \chi, A, B \in FS(\chi).$$

Definition 3.1.1 *An FS (or T1FS) A on χ is mapping $\mu_A : \chi \to U$ meaning that a real number $\mu_A(\mathrm{x}) \in U$ is the membership degree of such element $\mathrm{x} \in \chi$ in a FS A.*

Proposition 3.1.1 *(GORZAŁCZANY, 1988) The triple $(U, \cup_F, \cap_F)$ is a complete lattice (CL), where, for every $A, B \in U$, the union and intersection operations are respectively, given as:*

$$\mu_{A \cup_{FS} B}(x) = \max(\mu_A(\mathrm{x}), \mu_B(\mathrm{x})); \quad (1)$$

$$\mu_{A \cap_{FS} B}(x) = \min(\mu_A(\mathrm{x}), \mu_B(\mathrm{x})). \quad (2)$$

3.2 Automorphisms on U

According with (BUSTINCE; BURILLO; SORIA, 2003), a function $\psi : U \to U$ is an automorphism if it is continuous, strictly increasing and verifies the boundary conditions $\psi(0) = 0$ and $\psi(1) = 1$, i.e., if it is an increasing bijection on U, meaning that for each $x, y \in U$, if $x \leq y$, then $\psi(x) \leq \psi(y)$.

Automorphisms are closed under composition, i.e., denoting $Aut(U)$ the set of all automorphisms on U, if $\psi, \psi' \in A(U)$ then $\psi \circ \psi'(x) = \psi(\psi'(x)) \in Aut(U)$. In addition, the inverse ψ^{-1} of an automorphism $\psi \in Aut(U)$ is also an automorphism, meaning that $\psi^{-1} \in Aut(U)$.

By (BUSTINCE; BURILLO; SORIA, 2003), the action of an automorphism ψ on a function $f : U^n \to U$, denoted by f^ψ and named the ψ-conjugate of f is defined as,

$\forall(x_1,\ldots,x_n) \in U^n$:

$$f^{\psi}(x_1,\ldots,x_n) = \psi^{-1}(f(\psi(x_1),\ldots,\psi(x_n))). \quad (3)$$

3.3 Connectives in fuzzy logic

The axiomatic definition of fuzzy operators are considered below, including main properties related to negations, t-(co)norms and fuzzy implications.

3.3.1 Fuzzy negations and dual operators

A *fuzzy negation* (FN) is a function $N : U \to U$ verifies the properties:

N1 $N(0) = 1$ and $N(1) = 0$; and

N2 If $x \geq y$ then $N(x) \leq N(y)$, $\forall x, y \in U$.

In addition, fuzzy negations which also satisfy the involutive property, i.e.

N3 $N(N(x)) = x$, $\forall x \in U$.

are called strong fuzzy negations (SFN). The example of SFN is given by $N_S(x) = 1-x$ (Zadeh's standart negation).

Let N be a FN function and $f : U^n \longrightarrow U$ be real function. The N-dual function of f is denoted by $f_N : U^n \longrightarrow U$ and defined as:

$$f_N(x_1,\ldots,x_n) = N(f(N(x_1),\ldots,N(x_n))). \quad (4)$$

By Eq.(4), (f, f_N) is a pair of N-dual functions. If N is a SFN then, $(f_N)_N = f$, the N-dual function of f_N coincides with f. Thus, it is clear that f is a self-dual function.

3.3.2 Triangular norms and conorms

Two distinct classes of conjunctive and disjunctive aggregation functions called triangular norms (t-norms) and triangular conorms (t-conorms) provide interpretation for unions and intersections on $FS(\chi)$.

According with (KLEMENT; MESIAR; PAP, 2013), a t-norm and t-conorm are functions $T : U^2 \to U$ and $S : U^2 \to U$ satisfying, for all $x, y, z \in U$, the following properties:

T1: $T(x,y) = T(y,x)$;
T2: $T(x,T(y,z)) = T(T(x,y),z)$;
T3: If $y \leq z$ then $T(x,y) \leq T(x,z)$;
T4: $T(x,1) = x$.

S1: $S(x,y) = S(y,x)$;
S2: $S(x,S(y,z)) = S(S(x,y),z)$;
S3: If $y \leq z$ then $S(x,y) \leq S(x,z)$;
S4: $S(x,0) = x$.

A (right) left-continuous t-(co)norm $T : U^2 \to U$ for the first variable if it verifies for each countable (decreasing) increasing chain $(x_i)_{i\in\mathbb{N}}$ on $(U, \leq)$ the following condition:

T5: $\lim_{i\to\infty} T(\mathbf{x}_i, \mathbf{y}) = T(\lim_{i\to\infty} \mathbf{x}_i, \mathbf{y})$. **S5**: $\lim_{i\to\infty} S(\mathbf{x}_i, \mathbf{y}) = S(\lim_{i\to\infty} \mathbf{x}_i, \mathbf{y})$.

Analogously, a (right) left-continuous t-(co)norm $S : U^2 \to U$ for the second variable verifies for each countable (decreasing) increasing chain $(y_i)_{i\in\mathbb{N}}$ on $(U, \leq)$, and the following holds:

T6: $\lim_{i\to\infty} T(x, y_i) = T(x, \lim_{i\to\infty} y_i)$. **S6**: $\lim_{i\to\infty} S(x, y_i) = S(x, \lim_{i\to\infty} y_i)$.

Given the commutativity of $T(S), T5(S5)$ and $T6(S6)$ are equivalent.

Let N be a fuzzy negation on U. The mappings $S_N, T_N : U^2 \to U$ denoting the N-dual functions of a t-conorm S and a t-norm T, respectively, are defined as:

$$S_N(x, y) = N(S(N(x), N(y))); \qquad T_N(x, y) = N(T(N(x), N(y))). \tag{5}$$

Additionally, the N-dual function of a t-conorm S is a t-norm and vice-versa.

3.3.3 Fuzzy implications

The notion of a fuzzy implication extends the definition of a classic implication of Boolean logic, usually expressed by "$p \to q$" meaning that p is sufficient to deduct q. This extension refers to the fact that a fuzzy implication applied to the extremes of the unitary interval always coincides with the classic implication, but considers all other possibilities in this interval as arguments, modeling the uncertainty between totally true (1) and totally false (0).

A binary function $I : U^2 \to U$ is a *fuzzy implicator* if I meets the boundary conditions:

$\mathbf{I0}(a)$: $I(1, 1) = I(0, 1) = I(0, 0) = 1$;

$\mathbf{I0}(b)$: $I(1, 0) = 0$.

The set of all fuzzy implications is denoted by $I(U)$.

Definition 3.3.1 *(FODOR; ROUBENS, 1994, Definition 1.15) An implicator $I : U^2 \to U$ is a* fuzzy implication *if I also satisfies the conditions:*

$\mathbf{I1}$: *If $x \leq z$ then $I(x, y) \geq I(z, y)$ (first place antitonicity);*

$\mathbf{I2}$: *If $y \leq z$ then $I(x, y) \leq I(x, z)$ (second place isotonicity).*

Let $I(U)$ be the family of fuzzy implications on U.

Several reasonable properties may be required for fuzzy implications. The properties considered in this thesis are listed below:

$\mathbf{I3}:\ I(1,y)=y$ (left neutrality principle);

$\mathbf{I4}:\ I(x,1)=1$;

$\mathbf{I5}:\ I(x,I(y,z))=I(y,I(x,z))$ (exchange principle);

$\mathbf{I6}:\ I(x,y)=I(x,I(x,y))$ (iterative boolean-like law);

$\mathbf{I7}:\ I(x,N(x))=N(x)$ if N is a SFN;

$\mathbf{I8}:\ I(x,y)\geq y$;

$\mathbf{I9}:\ I(x,y)=I(N(y),N(x))$ if N is a SFN (the contrapositive symmetry);

$\mathbf{I10}:\ I(0,x)=1$, (dominance falsity);

$\mathbf{I11}:\ I(x,1)=1$, implies $x\leq y$;

$\mathbf{I12}:\ N(x)=I(x,0)$ is a SFN;

$\mathbf{I13}:\ I(x,y)=1$, if and only if $x\leq y$ (ordering property);

$\mathbf{I13}(a):\ I(x,y)=1$, implies $x\preceq y$;

$\mathbf{I14}:\ I(x,x)=1$, (identity principle).

Analogously, a fuzzy coimplicator J can be defined as a function $J:U^2\rightarrow U$, meeting the boundary conditions:

$\mathbf{J0}(a):\ J(1,1)=J(1,0)=J(0,0)=0$;

$\mathbf{J0}(b):\ J(0,1)=1$.

According to (RUIZ; TORRENS, 2004, Definition 7), while residual implicators can be viewed as a fuzzy generalization of the classical implication "$p\rightarrow q$", residual coimplicators generalize the classical coimplication "$q\nrightarrow p$".

By taking a fuzzy negation N, a fuzzy implication I gives rise to a fuzzy coimplication I_N which is the corresponding N-dual operator (REISER; BEDREGAL; REIS, 2014). Right afterwards, the duality is presented in next proposition:

Proposition 3.3.1 *(RUIZ; TORRENS, 2004) A function $(I_N)J_N:U^2\rightarrow U$ is a fuzzy (co)implication, if, and only if, there is a fuzzy coimplication (implication) $J(I){:}U^2\rightarrow U$ and strong fuzzy negation $N:U\rightarrow U$ such that $\forall x,y\in U^2$, any two following equalities are equivalent functions:*

$$I_N(x,y)=N(I(N(x),N(y)))\quad \textit{and}\quad J_N(x,y)=N(J(N(x),N(y))). \tag{6}$$

Most fuzzy implication operators and their corresponding interval extensions are based on two types of representations:

(i) the explicit representations defined in terms of aggregation operators, such as the classes of (S, N)-implications and QL-implications;

(ii) implicit representations, considering for instance R-implications.

The other ones which can not be classified in one of these two representations determine the class of implications referred to as A-implications, where relations with the aggregation operators are axiomatically defined based on algebraic properties (TURSKEN; KREINIVICH; YAGER, 1998). See e.g, Yager's implications and G_h functions which can not be naturally represented explicitly or implicitly (YAGER, 2004a).

Example 3.3.1 *Considering results in (BACZYŃSKI, 2003), four examples of representable fuzzy implications are the functions $I_{KD}, I_{RC}, I_{LK}, I_{WB} : U^2 \to U$ given by the expressions below:*

- $I_{KD}(x, y) = \max(1 - x, y)$;
- $I_{RC}(x, y) = 1 - x + xy$;
- $I_{LK}(x, y) = \min(1, 1 - x + y)$;
- $I_{WB}(x, y) = \begin{cases} 1, x \leq 1, \\ y, \textit{otherwise}. \end{cases}$

Moreover, results in (BACZYŃSKI; JAYARAM, 2007) support the following comparison:

$$I_{KD} \leq I_{RC} \leq I_{LK} \leq I_{WB}. \tag{7}$$

3.3.3.1 Action of an automorphis in an implication

Proposition 3.3.2 *In (BACZYŃSKI; DREWNIAK, 1999) fuzzy implications I,$J \in FI$ are conjugate if there exists an automorphism $\psi : U \to U$ such that $J = I^{\psi}$, where*

$$I^{\psi}(x, y) = \psi^{-1}(I(\psi(x), \psi(y))). \tag{8}$$

3.3.3.2 Main classes of fuzzy implications

This work focuses on the extension from U to $L_n(U)$ of explicit implications as (S, N)-implications and QL-implications, mainly in the implicit class of R-implications. So, underlying the foundations of the proposed extension, a brief description of such fuzzy classes are considered in the following.

3.3.3.3 (S, N)-implications

The class of (S, N)-implications generalizes the following classical equivalence: $p \to q \equiv \neg p \vee q$. The name (S, N)-implication was firstly introduced in the fuzzy logic framework by (TRILLAS; VALVERDE, 1981) since they are explicitly expressed by fuzzy negations and t-conorms on U. They are indicated by $I_{S,N} : U^2 \to U$ and defined by the following expressions:

$$I_{S,N}(x, y) = S(N(x), y), \forall x, y \in U, \tag{9}$$

when S is a t-conorm and N is a fuzzy negation.

When N is a strong fuzzy negation, then $I_{S,N}$ is a strong implication referred as (S, N)-implication. In the next proposition, a main characterization of (S, N)-implications is reported:

Proposition 3.3.3 *(BACZYŃSKI; JAYARAM, 2007, Theorem 1.7) Let $I : U^2 \to U$ be a fuzzy implication. I is an (S, N)-implication if and only if properties $\mathbf{I}1, \mathbf{I}2, \mathbf{I}3, \mathbf{I}5$ and $\mathbf{I}9$ are verified.*

The class of all (S, N)-implications are denoted by $\mathcal{C}_{I_{S,N}}(U)$.

3.3.3.4 QL-implications

Let S be a t-conorm, N be a strong fuzzy negation and T be a t-norm. A quantum-logic fuzzy implication, called QL-implication, is a function $I_{T,S,N} : U^2 \to U$ is defined by

$$I_{T,S,N}(x, y) = S(N(x), T(x, y)), \forall (x, y) \in U. \tag{10}$$

A result relating the QL-implications presented above with the properties w.r.t. the monotonicity and projection-functions is seen in the following proposition.

Proposition 3.3.4 *(BACZYŃSKI; JAYARAM, 2008, Proposition 2.6.2) Let $I : U^2 \to U$ be a fuzzy implication. If $I_{T,S,N}$ is a QL-operator. Then, the following holds:*

(i) *$I_{T,S,N}$ satisfies $\mathbf{I0}(a)$, $\mathbf{I0}(b)$ $\mathbf{I2}$, $\mathbf{I3}$ and $\mathbf{I10}$;*

(ii) $N_{I_{T,S,N}} = N$.

3.3.3.5 R-implications

Based on (BEDREGAL et al., 2009), by considering a t-norm $T : U^2 \to U$, the equation

$$I_T(x, y) = \sup\{z \in U \mid T(x, z) \leq y\}, \tag{11}$$

defines a fuzzy implication, called *R-implication* or *residuum* of T (BACZYŃSKI; JAYARAM, 2008a).

Thus, one can easily observe that an R-implication $I : U^2 \rightarrow U$ is well-defined if, and only if a t-norm $T : U^2 \rightarrow U$ is left-continuous (BUSTINCE; BURILLO; SORIA, 2003), meaning that

$$\lim_{n\to\infty} T(x_n, y) = T(\lim_{n\to\infty} x_n, y).$$

R-implications arise from the notion of residuum in Intuitionistic Logic (BACZYŃSKI; JAYARAM, 2008a) or, equivalently, from the notion of residue in the theory of lattice-ordered (FODOR, 1995), justifying the name “residuum of T”, since an R-implication satisfies the residuation condition when the underlying t-norm is left-continuous, meaning that

$$T(x, z) \leq y \text{ if and only if } I_T(x, y) \geq z. \tag{12}$$

Theorem 3.3.1 *(BACZYŃSKI; JAYARAM, 2008, Theorem 2.5.17.) For a function $I : U^2 \rightarrow U$ the following statements are equivalent:*

(i) *I is an R-implication generated from a left-continuous t-norm;*

(ii) *I satisfies $\mathbf{I2}$, $\mathbf{I5}$, $\mathbf{I13}$ and it is right-continuous with respect to the second variable.*

4 n-DIMENSIONAL FUZZY LOGIC: FUNDAMENTAL CONCEPTS

In Shang's investigations (SHANG; YUAN; LEE, 2010) and posterior studies of Bedregal et. al. (BEDREGAL et al., 2011; MEZZOMO et al., 2016), the n-dimensional fuzzy set theory is conceived as an extension of fuzzy set theory, extending main concepts from other fuzzy approaches as IVFS and interval-valued Atanassov intuitionistic fuzzy sets, since they can be seen as a particular case of fuzzy multisets (BEDREGAL et al., 2012). Even when the repetition of the membership degrees related to elements in an n-DFS is not considered, it can be defined as hesitant fuzzy set (HFS).

An n-dimensional fuzzy set considers tuples of size n whose components are valued in $U = [0;1]$ and ordered in increasing form, called n-dimensional intervals. Generally, these sets contribute in modeling situations involving decision-making where a given problem and an alternative, each n-dimensional interval represents the opinion of n specialists on the degree to which an alternative meets a given criterion or attribute for this problem.

By (BEDREGAL et al., 2011), the main idea in n-dimensional fuzzy set theory is to model several uncertainty levels in the memberships degrees, providing distinct interpretations, such as the following:

1. For $n > 1$, an n-dimensional interval can be obtained from an $n-1$-dimensional interval considering an uncertainty in one of its components. Thus, for example, given a membership degree a, if we have an uncertainty in the value, we can use a 2-dimensional interval (a_1, a_2) such that $a_1 \leq a \leq a_2$ (see the method in (JURIO et al., 2011)). Moreover, if again we have an uncertainty in a_1, then we could use a 3-dimensional interval (a_{1_1}, a_{1_2}, a_2) such that $a_{1_1} \leq a_1 \leq a_{1_2}$. If we have an uncertainty in a_2, we could use a 4-dimensional interval $(a_{1_1}, a_{1_2}, a_{2_1}, a_{2_2})$ such that $a_{2_1} \leq a_2 \leq a_{2_2}$, and so on.

2. Membership degrees which are given by n different evaluation processes and ordered in an increasing way.

3. When n is even, an n-dimensional interval can be seen as a chain of $\frac{n}{2}$ nested

intervals. Since the uncertainty measure of an interval is proportional to its length, an n-dimensional interval provides representation to distinct uncertainty levels on the membership degree. In this sense, the narrowest interval is an approximation of the membership degree reflecting the most optimistic uncertainty, whereas the broadest interval reflects the most pessimistic uncertainty.

The membership values of n-dimensional fuzzy sets are n-tuples of real numbers in the unit interval $U = [0,1]$, called n-dimensional intervals, which are ordered in an increasing order.

Let χ be a non empty set and $\mathbb{N}_n = \{1, 2, \ldots, n\}$. According to (SHANG; YUAN; LEE, 2010), Section 3, an n-dimensional fuzzy set $A_{L_n(U)}$ is expressed as

$$A_{L_n(U)} = \{(\mathrm{x}, \mu_{A_{L_n(U)}}(\mathrm{x}))\colon \forall \mathrm{x} \in \chi\}, \tag{13}$$

where $\forall \mathrm{x} \in \chi$, $\mu_{A_{L_n(U)}}(\mathrm{x}) = (x_1, x_2, \ldots, x_n)$ such that $x_1 \leq x_2 \leq \ldots \leq x_n$.

In another approach, an n-dimensional fuzzy set B over χ can also be given as

$$B_{L_n(U)} = \{(\mathrm{x}, \mu_{B1}(\mathrm{x}), \ldots, \mu_{Bn}(\mathrm{x})) : \mathrm{x} \in \chi\},$$

where all membership functions of B, denoted as $\mu_{Bi} : \chi \to U$, $\forall i \in \mathbb{N}_n$ verifies the condition $\mu_{B1}(\mathrm{x}) \leq \ldots \leq \mu_{Bn}(\mathrm{x})$, $\forall \mathrm{x} \in \chi$.

An $A_{L_n(U)}$ is a singleton n-dimensional fuzzy set if there exists an $\mathrm{x}' \in \chi$ such that $\mu_{A_{L_n(U)}}$ has the following representation:

$$\mu_{A_{L_n(U)}}(\mathrm{x}) = \begin{cases} /1/, & \text{if } \mathrm{x} = \mathrm{x}', \\ /0/, & \text{if } \mathrm{x} \neq \mathrm{x}'. \end{cases} \tag{14}$$

And, the set of all n-dimensional fuzzy set is denoted as $FS_n(\chi)$.

According to (BEDREGAL et al., 2011), the n-dimensional upper simplex is given as

$$L_n(U) = \{\mathbf{x} = (x_1, \ldots, x_n) \in U^n : x_1 \leq \ldots \leq x_n\}, \tag{15}$$

and an element $\mathbf{x} \in L_n(U)$ is called n-dimensional interval. For $i = \mathbb{N}_n$, the i-th projection of $L_n(U)$ is the function $\pi_i : L_n(U) \to U$ given by $\pi_i(x_1, \ldots, x_n) = x_i$.

A degenerate element $\mathbf{x} \in L_n(U)$ verifies the condition

$$\pi_i(\mathbf{x}) = \pi_j(\mathbf{x}), \forall i, j \in \mathbb{N}_n, \tag{16}$$

and will be denoted by $/x/$, for $x = \pi_i(\mathbf{x})$.

4.1 Partial orders on $L_n(U)$

In this section, based on previous results presented in (BEDREGAL et al., 2011), the main concepts of partial orders and their extensions based on admissible linear orders are discussed.

By considering the natural extension of the usual order $\leq$ from U to $L_n(U)$ then, for each $\mathbf{x}, \mathbf{y} \in L_n(U)$, the following holds:

$$\mathbf{x} \leq \mathbf{y} \;\Leftrightarrow\; \pi_i(\mathbf{x}) \leq_U \pi_i(\mathbf{y}), \;\; \forall i \in \mathbb{N}_n. \tag{17}$$

Moreover, for all $\mathbf{x}, \mathbf{y} \in L_n(U)$, the supremum and infimum with respect to $\leq$-order is given as:

$$\mathbf{x} \vee \mathbf{y} = (\max(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, \max(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))); \tag{18}$$

$$\mathbf{x} \wedge \mathbf{y} = (\min(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, \min(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))). \tag{19}$$

By (GIERZ et al., 2003), $\mathcal{L}_n(U) = (L_n(U), \vee, \wedge, /0/, /1/)$ is a distributive complete lattice, which is continuous, with $/0/$ and $/1/$ being their bottom and top elements, respectively. Since its directed subsets has a supremum, $(L_n(U), \leq)$ is a distributive complete partial-ordered lattice (DCPO). We can easily observe that $L_1(U) = U$ and $L_2(U)$ reduces to the usual lattice $L(U)$ of all the closed subintervals on U.

Additionally, the following relation is also considered

$$\mathbf{x} \preceq \mathbf{y} \Leftrightarrow \mathbf{x} = \mathbf{y} \text{ or } \pi_n(\mathbf{x}) \leq \pi_1(\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in L_n(U), \tag{20}$$

since it is related to partial orders on $L_n(U)$, one can easily observe that $\preceq$ is more restrictive than $\leq$, meaning that

$$\mathbf{x} \preceq \mathbf{y} \Rightarrow \mathbf{x} \leq \mathbf{y}.$$

4.2 Continuity on $L_n(U)$

The continuity of functions is based on the particular topology of both domain and codomain. There are topological spaces which can be derived from metrics and, in this case, the continuity of functions has an equivalent definition based on metrics. In this section we will consider main properties and relevant notions of continuity in metric spaces. For more thorough studies see Dugundji (DUGUNDJI, 1966).

Based on results from Proposition 3.1 in (MEZZOMO; MILFONT; BEDREGAL, 2018), the function $d_M^n : L_n(U) \times L_n(U) \to \mathbb{R}^+$ defined by

$$d_M^n(\mathbf{x}, \mathbf{y}) = \max(|\pi_1(\mathbf{x}) - \pi_1(\mathbf{y})|, \ldots, |\pi_n(\mathbf{x}) - \pi_n(\mathbf{y})|), \tag{21}$$

is a metric on $L_n(U)$, called the n-dimensional interval Moore-metric on $L_n(U)$.

Remark 4.2.1 *When the natural immersion of $L_n(U)$ in U^n is considered, the n-dimensional interval Moore-metric coincides with the Chebyshev-metric (MENDELSON, 1990). Moreover, d_M^1 in Eq.(21) is the usual distance on real numbers restricted to U and d_M^2 in Eq.(21) is the Moore-metric (DIMURO et al., 2011).*

Since d_M^n is a metric, a continuity notion for n-dimensional unary functions is also verified. Thus, in order to study the continuity of n-dimensional functions of arbitrary m-arity, we need to consider a corresponding metric on $(L_n(U))^m$.

Proposition 4.2.1 *Let $d_M^{n,m} : (L_n(U))^m \times (L_n(U))^m \to \mathbb{R}^+$ be a function defined, for each $\vec{\mathbf{x}} = (\mathbf{x_1}, \ldots, \mathbf{x_m}), \vec{\mathbf{y}} = (\mathbf{y_1}, \ldots, \mathbf{y_m}) \in (L_n(U))^m$, as follows:*

$$d_M^{n,m}(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = \max(d_M^n(\mathbf{x_1}, \mathbf{y_1}), \ldots, d_M^n(\mathbf{x_m}, \mathbf{y_m})). \tag{22}$$

Then, $d_M^{n,m}$ is a metric on $(L_n(U))^m$.

Proof: Clearly $d_M^{n,m}$ is symmetric and since d_M^n is a metric, for $\vec{\mathbf{x}}, \vec{\mathbf{y}}, \vec{\mathbf{z}} \in (L_n(U))^m$ the next two conditions are verified:
(i) $d_M^{n,m}(\vec{\mathbf{x}}, \vec{\mathbf{y}}) = 0$ if, and only if, $d_M^n(\mathbf{x_i}, \mathbf{y_i}) = 0$ for each $i \in \mathbb{N}_n$ if, and only if, $\mathbf{x_i} = \mathbf{y_i}$ for each $i \in \mathbb{N}_n$ if, and only if, $\vec{\mathbf{x}} = \vec{\mathbf{y}}$;
(ii) $\forall i \in \mathbb{N}_n$, when $d_M^n(\mathbf{x_i}, \mathbf{z_i}) \leq d_M^n(\mathbf{x_i}, \mathbf{y_i}) + d_M^n(\mathbf{y_i}, \mathbf{z_i})$ then we obtain that $\max(d_M^n(\mathbf{x_1}, \mathbf{z_1}), \ldots, d_M^n(\mathbf{x_m}, \mathbf{z_m})) \leq \max(d_M^n(\mathbf{x_1}, \mathbf{y_1}), \ldots, d_M^n(\mathbf{x_m}, \mathbf{y_m})) + \max(d_M^n(\mathbf{y_1}, \mathbf{z_1}), \ldots, d_M^n(\mathbf{y_m}, \mathbf{z_m}))$. Therefore, the following holds $d_M^{n,m}(\vec{\mathbf{x}}, \vec{\mathbf{z}}) \leq d_M^{n,m}(\vec{\mathbf{x}}, \vec{\mathbf{y}}) + d_M^{n,m}(\vec{\mathbf{y}} = \vec{\mathbf{z}})$. Concluding, Proposition 4.2.1 is verified. □

Let $(d_M^{n,m}, d_M^n)$-continuous be a function $F : (L_n(U))^m \to L_n(U)$. It is called as Moore-continuous. And, in the following, the convergence sequences and limits on the set of real intervals U^2 are extended to $L_n(U)$.

Definition 4.2.1 *An n-dimensional interval sequence (n-DIS) is a function $f : \mathbb{N} \to L_n(U)$ meaning that $f(i) = \mathbf{x}_i, \forall i \in \mathbb{N}$ and usually denoted by $(\mathbf{x}_i)_{i \in \mathbb{N}}$.*

Definition 4.2.2 *An n-DIS $(\mathbf{x}_i)_{i \in \mathbb{N}}$ converge to $\mathbf{a} \in L_n(U)$, denoted by $\mathbf{x}_i \to \mathbf{a}$ or $\lim_{n \to \infty} \mathbf{x}_i = \mathbf{a}$ whenever the following holds:*

$$\forall \varepsilon > 0, \exists n_0 \geq 0 \text{ such that } d_M^n(\mathbf{x}_i, \mathbf{a}) < \varepsilon, \forall n > n_0.$$

In addition, when, $\forall k \in \mathbb{N}_{n-1}$, the following is also verified verified:

(i) $\mathbf{x}_k \leq \mathbf{x}_{k+1}$, then $(\mathbf{x}_i)_{i \in \mathbb{N}}$ is an increasing n-DIS; and

(ii) $\mathbf{x}_k \geq \mathbf{x}_{k+1}$, then $(\mathbf{x}_i)_{i \in \mathbb{N}}$ is an decreasing n-DIS.

A function $\mathcal{F} : (L_n(U))^2 \to L_n(U)$ is a (right) left-continuous for the first variable if it verifies for each countable(decreasing) increasing chain $(\mathbf{x}_i)_{i\in\mathbb{N}}$ on $(L_n(U), \leq)$, meaning that the following holds:

$$\mathcal{RC}: \lim_{i\to\infty} \mathcal{F}(\mathbf{x}_i, \mathbf{y}) = \mathcal{F}(\lim_{i\to\infty} \mathbf{x}_i, \mathbf{y}). \tag{23}$$

Analogously, a (right) left-continuous n-DIS $\mathcal{F} : (L_n(U))^2 \to L_n(U)$ for the second variable verifies for each countable(decreasing) increasing chain $(\mathbf{y}_i)_{i\in\mathbb{N}}$ on $(L_n(U), \leq)$, meaning that the following holds:

$$\mathcal{LC}: \lim_{i\to\infty} \mathcal{F}(\mathbf{x}, \mathbf{y}_i) = \mathcal{F}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i). \tag{24}$$

Proposition 4.2.2 *An n-DIS $(\mathbf{x}_i)_{n\,\in\,\mathbb{N}}$ converge to $\mathbf{a} \in L_n(U)$ if, and only if, $\pi_k((\mathbf{x}_i)_{i\in\mathbb{N}}) = (\pi_k(\mathbf{x}_i))_{i\in\mathbb{N}}$ also converge to $\pi_k(\mathbf{a})$, for each $k \in \mathbb{N}_n$.*

Proof: Straightforward. □

4.3 Automorphisms and conjugate functions on $L_n(U)$

This section reports conditions under which n-dimensional fuzzy automorphisms are considered to obtain other conjugate-operators on $L_n(U)$, preserving their main properties.

Definition 4.3.1 *(BEDREGAL et al., 2012) A function $\varphi : L_n(U) \to L_n(U)$ is an n-dimensional automorphism if φ is bijective and the following condition is satisfied:*

$$\mathbf{x} \leq \mathbf{y} \Leftrightarrow \varphi(\mathbf{x}) \leq \varphi(\mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in L_n(U). \tag{25}$$

The sets of all automorphims on $L_n(U)$ and U are denoted by $Aut(L_n(U))$ and $Aut(U)$, respectively.

Proposition 4.3.1 *From (BEDREGAL et al., 2012, Theorem 3.4), let $\varphi : L_n(U) \to L_n(U)$. Then, $\varphi \in Aut(L_n(U))$ if, and only if, there exists $\psi \in Aut(U)$ such that*

$$\varphi(\mathbf{x}) = (\psi(\pi_1(\mathbf{x})), \ldots, \psi(\pi_n(\mathbf{x}))). \tag{26}$$

So, we will denote φ by $\widetilde{\psi}$, for all $\mathbf{x} = (x_1, \ldots, x_n) \in L_n(U)$, the Eq.(26) can be expressed as

$$\widetilde{\psi}(\mathbf{x}) = (\psi(\pi_1(\mathbf{x})), \ldots, \psi(\pi_n(\mathbf{x}))). \tag{27}$$

In (BEDREGAL; MEZZOMO; REISER, 2018, Corollary 4.1), for $\varphi \in Aut(L_n(U))$, then φ is also continuous, strictly increasing, $\varphi(/0/) = /0/$ and $\varphi(/1/) = /1/$.

Proposition 4.3.2 *(BEDREGAL et al., 2012, Proposition 3.4) When $\psi \in Aut(U)$, then $\widetilde{\psi^{-1}} = \widetilde{\psi}^{-1}$.*

Given a function $F : L_n(U)^n \to L_n(U)$ and an n-dimensional automorphism φ, the action of φ to F is the function $F^{\varphi} : L_n(U)^n \to L_n(U)$ defined by

$$F^{\varphi}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \varphi^{-1}(F(\varphi(\mathbf{x}_1), \ldots, \varphi(\mathbf{x}_n))). \tag{28}$$

F^{φ} is said the conjugate of F. We will denote $F^{(\varphi)} = \varphi \circ F^{\varphi}$ or equivalently:

$$F^{\varphi}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = F(\varphi(\mathbf{x}_1), \ldots, \varphi(\mathbf{x}_n)). \tag{29}$$

4.4 Negations and dual functions on $L_n(U)$

Preliminaries researches on n-dimensional fuzzy negations were already considered, see results in (BEDREGAL; MEZZOMO; REISER, 2018) and (MEZZOMO; BEDREGAL; REISER, 2017). In such families of n-DFS, fuzzy preference intensity in the classification model was arranged according to the basic preference attitudes is already explored (AMO et al., 2004) based on n-dimensional fuzzy negations.

As conceived in (BEDREGAL et al., 2012), the notion of fuzzy negation was extended to $L_n(U)$ and its main concepts are reported below.

Definition 4.4.1 *A function $\mathcal{N} : L_n(U) \to L_n(U)$ is an n-dimensional fuzzy negation (n-DN) if it satisfies:*

$\mathcal{N}$1: $\mathcal{N}(/0/) = /1/$ and $\mathcal{N}(/1/) = /0/$;

$\mathcal{N}$2: If $\mathbf{x} \leq \mathbf{y}$ then $\mathcal{N}(\mathbf{x}) \geq \mathcal{N}(\mathbf{y})$.

A n-DN $\mathcal{N}$ is strict if it is continuous and $\mathcal{N}(\mathbf{x}) < \mathcal{N}(\mathbf{y})$ when $\mathbf{y} < \mathbf{x}$. Moreover, when an n-DN $\mathcal{N}$ satisfies the following condition:

$\mathcal{N}$3: : $\mathcal{N}(\mathcal{N}(\mathbf{x})) = \mathbf{x}$ for each $\mathbf{x} \in L_n(U)$ (Involution),

it is called a strong n-DN.

Studies on n-DN extends the preliminary studies on representability of fuzzy negations (FN), preserving their main properties.

According to (BEDREGAL et al., 2012, Proposition 3.1), if $N_1, \ldots, N_n$ are fuzzy negations such that $N_1 \leq \ldots \leq N_n$, then $\widetilde{N_1 \ldots N_n} : L_n(U) \to L_n(U)$ is a representable n-DN given by

$$\widetilde{N_1 \ldots N_n}(\mathbf{x}) = (N_1(\pi_n(\mathbf{x})), \ldots, N_n(\pi_1(\mathbf{x}))). \tag{30}$$

When $N = N_1 = \ldots = N_n$, $\widetilde{N_1 \ldots N_n}$ is denoted as $\widetilde{N}$.

Example 4.4.1 *Considering $N_{D1}, N_{D2} : U \to U$ as fuzzy negations respectively given as follows:*

$$N_{D1}(x) = \begin{cases} 1, & \textit{if } x = 0, \\ 0, & \textit{otherwise}; \end{cases} \qquad N_{D2}(x) = \begin{cases} 0, & \textit{if } x = 1, \\ 1, & \textit{otherwise}; \end{cases}$$

and, $N_S, N_K, N_R : U \to U$ given as $N_S(x) = 1 - x$, $N_K(x) = 1 - \sqrt{x}$ and $N_R(x) = 1 - x^2$. From Eq.(30), the following is verified:
(i) $\widetilde{N_{D1}, N_R, N_S, N_K, N_{D2}} : L_n(U) \to L_n(U)$ is a representable n-DN;
(ii) $\widetilde{N_{D1}}, \widetilde{N_S}, \widetilde{N_K}, \widetilde{N_R}, \widetilde{N_{D2}} : L_n(U) \to L_n(U)$ are the n-dimensional interval extensions of the above fuzzy negations.

Proposition 4.4.1 *(BEDREGAL; MEZZOMO; REISER, 2018) Let $\mathcal{N}$ be an n-DN. Then, the function $N_i : U \to U$ is a fuzzy negation defined by*

$$N_i(x) = \pi_i(\mathcal{N}(/x/)), \forall i \in \mathbb{N}_n, x \in U. \tag{31}$$

Theorem 4.4.1 *(MEZZOMO; MILFONT; BEDREGAL, 2018, Theorem 3.1) Let $N_1, \ldots, N_n$ be fuzzy negations such that $N_1 \leq \ldots \leq N_n$. The n-DN $\widetilde{N_1 \ldots N_n} : L_n(U) \to L_n(U)$ given by Eq.(30) is Moore continuous if, and only if, every N_i, $i \in \mathbb{N}_n$, is continuous.*

Based on results from (MEZZOMO; MILFONT; BEDREGAL, 2018, Corollary 3.1) each strong n-DN is Moore continuous.

Let $\mathcal{N}$ be a strong n-DN and $\mathcal{F} : (L_n(U))^n \to L_n(u)$ be a function. The $\mathcal{N}$-dual of $\mathcal{F}$ is the function $\mathcal{F}_{\mathcal{N}} : (L_n(U))^n \to L_n(U)$ given as

$$\mathcal{F}_{\mathcal{N}}(\mathbf{x}_1, \ldots, \mathbf{x}_n) = \mathcal{N}(\mathcal{F}(\mathcal{N}(\mathbf{x}_1), \ldots, \mathcal{N}(\mathbf{x}_n))). \tag{32}$$

Concluding this section, according to (BEDREGAL; MEZZOMO; REISER, 2018, Proposition 4.2), let $\varphi \in Aut(L_n(U))$. $\mathcal{N}$ is an n-dimensional (strict, strong) fuzzy negation if, and only if, $\mathcal{N}^{\varphi}$ is an n-dimensional (strict, strong) fuzzy negation such that, for all $\mathbf{x} \in L_n(U)$, the following holds:

$$\mathcal{N}^{\varphi}(\mathbf{x}) = \varphi^{-1}(\mathcal{N}(\varphi(\mathbf{x}))). \tag{33}$$

4.5 Aggregations functions and admissible orders on $L_n(U)$

According to (DETYNIECKI, 2001), aggregation and fusion of information are basic initial concerns for systems ranging from image processing to DM, from pattern recognition to machine learning. Aiming at using different information (provided by various sources), in order to reach a conclusion or decision.

By (BEDREGAL; MEZZOMO; REISER, 2018), a k-ary n-dimensional aggregation function (n-DA) $\mathcal{M} : L_n(U)^k \to L_n(U)$ is a function satisfying, for all $(\mathbf{x}_1, \ldots, \mathbf{x}_k), (\mathbf{y}_1, \ldots, \mathbf{y}_k) \in L_n(U)^k$, the conditions:

$\mathcal{A}1$**:** $\mathcal{M}(/0/, \ldots, /0/) = /0/$ and $\mathcal{M}(/1/, \ldots, /1/) = /1/$;

$\mathcal{A}2$**:** $\mathbf{x}_i \leq \mathbf{y}_i$ for all $i \in \mathbb{N}_k \Rightarrow \mathcal{M}(\mathbf{x}_1, \ldots, \mathbf{x}_k) \leq \mathcal{M}(\mathbf{y}_1, \ldots, \mathbf{y}_k)$.

When $n = 1$ then $L_1(U) = U$ and therefore, each 1-DA is a k-ary aggregation function $M : U^k \to U$.

Example 4.5.1 *By Eq.(19),* $i \in N_n$ *and* $j \in N_k$ *we will denote* $\pi_i(\mathbf{x}_j)$*,* $\mathbf{x}_{ij}$*, The arithmetic mean aggregation and minimum operators* $AM\wedge : (L_n(U))^k \to L_n(U)$ *which are given as:*

$$AM \wedge (\mathbf{x}_1, \ldots, \mathbf{x}_k) = \frac{1}{k}\left(\sum_{i=1}^{k} x_{i1}, \ldots, \sum_{i=1}^{k} x_{in}\right); \tag{34}$$

$$\bigwedge(\mathbf{x}_1, \ldots, \mathbf{x}_k) = \wedge_{i=1}^{k}\mathbf{x}_i = (\min_{i=1}^{n} x_{1i}, \ldots, \min_{i=1}^{n} x_{ki}). \tag{35}$$

4.5.1 Admissible linear orders on $L_n(U)$

An admissible linear order $\preceq$ can be considered in order to amplify the set of comparable pairs of elements in $(L_n(U), \leq)$. In fact, all the pairs of elements in $(L_n(U), \preceq)$ are compared.

Definition 4.5.1 *(ZAPATA et al., 2017) Let* $(L_n(U), \leq)$ *be a poset. The order* $\preceq$ *is called an admissible order on* $(L_n(U), \preceq)$*, if both conditions hold:*

(i) $\preceq$ *is a linear order on* $L_n(U)$*; and*

(ii) $\preceq$ *refines* $\leq$*, meaning that, for all* $[a_i, \ldots, a_n], [b_i, \ldots, b_n] \in L_n(U)$ *we have that*

$$[a_i, \ldots, a_n] \leq [b_i, \ldots, b_n] \Rightarrow [a_i, \ldots, a_n] \preceq [b_i, \ldots, b_n].$$

So, an order $\preceq$ is admissible on $L_n(U)$ when it is linear and refines the order $\leq$ on $L_n(U)$.

Example 4.5.2 *Based on (ZAPATA et al., 2017), extensions of admissible linear order from* $L_2(U)$ *on* $L_n(U)$ *are presented. Taking* n*-dimensional intervals*

$$\mathbf{a} = (a_1, a_2, ..., a_n), \mathbf{b} = (b_1, b_2, ..., b_n) \in L_n(U),$$

we have the following three admissible linear orders:

(1) *the lexicographic order $\preceq_{Lex1}$: $\mathbf{a} \preceq_{Lex1} \mathbf{b}$ if the ith component of $\mathbf{a}$ is strictly less than the ith component of $\mathbf{b}$, whereas $a_j = b_j$, $\forall j < i, j, i \in \mathbb{N}_n$, following that:*

$$\mathbf{a} \preceq_{Lex1} \mathbf{b} \quad \Leftrightarrow \quad \begin{cases} a_1 < b_1; \\ a_1 = b_1 \ \wedge \ a_2 < b_2; \\ \quad \vdots \\ a_1 = b_1 \wedge \ldots \wedge \wedge a_{n-1} = b_{n-1} \wedge a_n < b_n. \end{cases}$$

(2) *the reverse lexicographic order $\preceq_{Lex2}$: $\mathbf{a} \preceq_{Lex2} \mathbf{b}$ if the ith component of $\mathbf{a}$ is strictly less than the ith component of $\mathbf{b}$, whereas: $a_j = b_j, \forall j > i, j, i \in \mathbb{N}_n$, following that:*

$$\mathbf{a} \preceq_{Lex2} \mathbf{b} \quad \Leftrightarrow \quad \begin{cases} a_n < b_n; \\ a_n = b_n \ \wedge \ a_{n-1} < b_{n-1}; \\ \quad \vdots \\ a_n = b_n \wedge \ldots \wedge a_2 = b_2 \wedge a_1 < b_1. \end{cases}$$

(3) *the Xu-Yager's admissible linear order $\preceq_{YX}$ defined as follows:*

$$\mathbf{a} \preceq_{XY} \mathbf{b} \quad \Leftrightarrow \quad \begin{cases} a_2 + a_1 < b_2 + b_1; \\ a_2 + a_1 = b_2 + b_1 \wedge a_2 - a_1 < b_2 - b_1; \\ \quad \vdots \\ a_2 + a_1 = b_2 + b_1 \wedge \ldots \wedge a_n + a_{n-1} \\ \qquad\qquad = b_n + b_{n-1} \wedge a_n - a_{n-1} \leq b_n - b_{n-1}. \end{cases}$$

In addition, according to (DE MIGUEL et al., 2017), we can define admissible linear orders on $L_n(U)$ by taking into account a sequence $\mathcal{M} = (M_1, \ldots, M_n)$ of aggregations functions $M_i : U^n \to U$, for each $i \in \mathbb{N}_n$.Thus, for $\mathbf{x}, \mathbf{y} \in L_n(U)$, the $\sqsubseteq_{\mathcal{M}}$-order relation is defined as follows:

(i) $\mathbf{x} \sqsubset_{\mathcal{M}} \mathbf{y}$ if, and only if, there exists $k \in \mathbb{N}_n$ such that $M_j(\mathbf{x}) = M_j(\mathbf{y})$, $\forall j \in \mathbb{N}_{k-1}$ and $M_k(\mathbf{x}) < M_k(\mathbf{y})$;

(ii) $\mathbf{x} \sqsubseteq_{\mathcal{M}} \mathbf{y}$ if, and only if, $\mathbf{x} \sqsubset_{\mathcal{M}} \mathbf{y}$ or $\mathbf{x} = \mathbf{y}$.

The next results provide the conditions under which we can obtain an $\sqsubseteq_{\mathcal{M}}$-order relation enabling us to compare all n-dimensional intervals on $L_n(U)$.

Proposition 4.5.1 *(DE MIGUEL et al., 2017, Proposition 1) Let $\mathcal{M} = (M_1 \ldots M_n)$ be a sequence of n aggregations functions $M_i : U^n \to U$, for each $i \in \mathbb{N}_n$. The $\sqsubseteq_{\mathcal{M}}$-order* *relation on $L_n(U)$ is admissible if, and only if, for each $\mathbf{x}, \mathbf{y} \in L_n(U)$, it holds that:*

$$M_i(\mathbf{x}) = M_i(\mathbf{y}), \forall i \in \mathbb{N}_n \Leftrightarrow \mathbf{x} = \mathbf{y}.$$

Proposition 4.5.2 *(DE MIGUEL et al., 2017, Proposition 2) Let $\mathcal{M} = (M_1 \dots M_n)$ be a sequence of n aggregations functions $M_i : U^n \to U$ given as $M_i(\mathbf{x}) = \alpha_{i1}(\pi_1(\mathbf{x})) + \dots + \alpha_{in}(\pi_n(\mathbf{x}))$, whenever $\alpha_{i1} + \dots + \alpha_{in} = 1$ and $0 \leq \alpha_{ij} \leq 1$, for each $i, j \in \mathbb{N}_n$. The $\sqsubseteq_{\mathcal{M}}$-order on $L_n(U)$ is admissible if, and only if, the corresponding matrix $[M] = (\alpha_{ij})_{n \times n}$ is regular.*

Example 4.5.3 *Let $M_1, M_2, M_3, M_4 : U^4 \to U$ be aggregations defined as follows:*

$$\begin{aligned}
M_1(\mathbf{x}) &= 0.25x_1 + 0.25x_2 + 0.25x_3 + 0.25x_4;\\
M_2(\mathbf{x}) &= 0.5x_1 + 0.15x_2 + 0.15x_3 + 0.2x_4;\\
M_3(\mathbf{x}) &= 0.2x_1 + 0.2x_2 + 0.3x_3 + 0.3x_4;\\
M_4(\mathbf{x}) &= 0.1x_1 + 0.4x_2 + 0.1x_3 + 0.4x_4.
\end{aligned}$$

Under the conditions of Proposition 4.5.2, $[M] = (\alpha_{ij})_{4 \times 4}$ is a regular matrix meaning that $M_i(\mathbf{x}) = M_i(\mathbf{y}), \forall i \in \mathbb{N}_4 \Leftrightarrow \mathbf{x} = \mathbf{y}$. Therefore, the $\sqsubseteq_{\mathcal{M}}$-order on $L_4(U)$ is admissible for $M = \{M_1, M_2, M_3, M_4\}$.

4.5.2 Triangular norms and triangular conorms on $L_n(U)$

In (MEZZOMO; BEDREGAL; REISER, 2017), the notion of t-conorms on U was extended to t-conorms on $L_n(U)$ as follows:

Definition 4.5.2 *(BEDREGAL et al., 2012, Definition 3.4) An n-dimensional t-norm and n-dimensional t-conorm are functions $\mathcal{T} : (L_n(U))^2 \to L_n(U)$ and $(\mathcal{S}) : (L_n(U))^2 \to L_n(U)$ satisfying, for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$, the conditions: it has $/1/$ as the neutral element for t-norm and $/0/$ as the neutral element for t-conorm together with it is commutative, associative and monotonic function with respect to the partial order $\leq$. It means that the properties below are verified:*

$\mathcal{T}1$: $\mathcal{T}(\mathbf{x}, /1/) = \mathbf{x}$;	*$\mathcal{S}1$: $\mathcal{S}(\mathbf{x}, /0/) = \mathbf{x}$;*
$\mathcal{T}2$: $\mathcal{T}(\mathbf{x}, \mathbf{y}) = \mathcal{T}(\mathbf{y}, \mathbf{x})$;	*$\mathcal{S}2$: $\mathcal{S}(\mathbf{x}, \mathbf{y}) = \mathcal{S}(\mathbf{y}, \mathbf{x})$;*
$\mathcal{T}3$: $\mathcal{T}(\mathbf{x}, \mathcal{T}(\mathbf{y}, \mathbf{z})) = \mathcal{T}(\mathcal{T}(\mathbf{x}, \mathbf{y}), \mathbf{z})$;	*$\mathcal{S}3$: $\mathcal{S}(\mathbf{x}, S(\mathbf{y}, \mathbf{z})) = \mathcal{S}(\mathcal{S}(\mathbf{x}, \mathbf{y}), \mathbf{z})$;*
$\mathcal{T}4$: if $\mathbf{x} \leq \mathbf{y}$, then $\mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathcal{T}(\mathbf{y}, \mathbf{z})$.	*$\mathcal{S}4$: if $\mathbf{y} \leq \mathbf{z}$, then $\mathcal{S}(\mathbf{x}, \mathbf{y}) \leq \mathcal{S}(\mathbf{x}, \mathbf{z})$.*

Another property verified by some t-(co)norm on $L_n(U)$ is as law of excluded middle (LEM) following:

LEM: $\mathcal{T}(\mathcal{N}(\mathbf{x}), \mathbf{x}) = /0/$. $\qquad$ $\mathcal{S}(\mathcal{N}(\mathbf{x}), \mathbf{x}) = /1/$.

4.5.3 Representability and duality of n-DT and n-DS on $L_n(U)$

According to (BEDREGAL et al., 2012), the conditions under which an n-dimensional t-(co)norm on $\mathcal{L}_n = (L_n(U), \vee, \wedge, /0/, /1/)$ can be obtained from a finite subset of t-(co)norms on U are reported in the following.

Theorem 4.5.1 *(MEZZOMO; BEDREGAL; REISER, 2017, Theorem 2.1) If there exist t-(co)norms $T_1, \ldots, T_n (S_1, \ldots, S_n)$ such that $T_1 \leq \ldots \leq T_n (S_1 \leq \ldots \leq S_n)$, then the function $\widetilde{T_1 \ldots T_n} (\widetilde{S_1 \ldots S_n}) : (L_n(U))^2 \rightarrow L_n(U)$, is an n-dimensional triangular (co)norm, indicated as n-DT (n-DS) and defined by*

$$\widetilde{T_1 \ldots T_n}(\mathbf{x}, \mathbf{y}) = (T_1(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, T_n(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))); \tag{36}$$

$$\widetilde{S_1 \ldots S_n}(\mathbf{x}, \mathbf{y}) = (S_1(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, S_n(\pi_n(\mathbf{x}), \pi_n(\mathbf{y}))). \tag{37}$$

As a consequence of Theorem 4.5.1, Eqs.(36) and (37) defining n-DT (n-DS) $\mathcal{T}$ ($\mathcal{S}$) can be reduced to the following equations

$$\widetilde{T_1 \ldots T_n}(\mathbf{x}_1, \mathbf{x}_2) = (T_1(x_{11}, x_{21}), \ldots, T_n(x_{1n}, x_{2n})); \tag{38}$$

$$\widetilde{S_1 \ldots S_n}(\mathbf{x}_1, \mathbf{x}_2) = (S_1(x_{11}, x_{21}), \ldots, S_n(x_{1n}, x_{2n})), \tag{39}$$

is called a representable n-DT (n-DS) $\mathcal{T}$ ($\mathcal{S}$).

More, when $T_1 = \ldots = T_n = T$ and $S_1 = \ldots = S_n = S$, then n-dimensional t-norm $\widetilde{T_1 \ldots T_n}$ and t-conorm $\widetilde{S_1 \ldots S_n}$ will be respectively denoted by $\widetilde{T}$ and $\widetilde{S}$.

Example 4.5.4 *According to (BEDREGAL et al., 2012), an example of an n-dimensional t-norm which is not t-representable is*

$$\mathcal{T}(\mathbf{x}, \mathbf{y}) = (\min(\pi_1(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, \min(\pi_{n-1}(\mathbf{x}), \pi_{n-1}(\mathbf{y})),$$
$$\max(\min(\pi_{n-1}(\mathbf{x}), \pi_n(\mathbf{y}), \ldots \min(\pi_n(\mathbf{x}), \pi_{n-1}(\mathbf{y})))).$$

Based on the both aggregation operators $\wedge$ and $\vee$, additional properties of n-DT and n-DS can be considered:

$\mathcal{T}5$. $\mathcal{T}(\mathbf{x}, \mathbf{y} \vee \mathbf{z}) = \mathcal{T}(\mathbf{x}, \mathbf{y}) \vee \mathcal{T}(\mathbf{x}, \mathbf{z})$; $\mathcal{S}5$. $\mathcal{S}(\mathbf{x}, \mathbf{y} \wedge \mathbf{z}) = \mathcal{S}(\mathbf{x}, \mathbf{y}) \wedge \mathcal{S}(\mathbf{x}, \mathbf{z})$.

Example 4.5.5 *Two representable n-dimensional aggregation operators on $L_n(U)$ w.r.t. the order $\leq$ on $L_n(U)$, are the following:*

(a) $\wedge(\mathbf{x_1}, \ldots, \mathbf{x_m}) = (\min(\pi_\mathbf{1}(\mathbf{x_1}), \ldots, \pi_\mathbf{1}(\mathbf{x_m})), \ldots, \min(\pi_\mathbf{n}(\mathbf{x_1}), \ldots, \pi_\mathbf{n}(\mathbf{x_m})))$;

(b) $\vee(\mathbf{x_1}, \ldots, \mathbf{x_m}) = (\max(\pi_\mathbf{1}(\mathbf{x_1}), \ldots, \pi_\mathbf{1}(\mathbf{x_m})), \ldots, \max(\pi_\mathbf{n}(\mathbf{x_1}), \ldots, \pi_\mathbf{n}(\mathbf{x_m})))$.

Proposition 4.5.3 *(ZANOTELLI; REISER; BEDREGAL, 2018a) Let $\mathcal{S}$ ($\mathcal{T}$) be a representable n-DS (n-DT) and $\widetilde{N}$ be a strong n-DN (Sn-DN) with $T_1, \ldots, T_n$ and $S_1, \ldots, S_n$ as their representants. The function $\mathcal{S}_{\widetilde{N}}(\mathcal{T}_{\widetilde{N}}) : (L_n(U))^2 \to L_n(U)$ given as*

$$\mathcal{S}_{\widetilde{N}}(\mathbf{x}_1, \mathbf{x}_2) = \widetilde{S_1 \ldots S_n}_{\widetilde{N}}(\mathbf{x}_1, \mathbf{x}_2) = \widetilde{S_{n_N} \ldots S_{1_N}}(\mathbf{x}_1, \mathbf{x}_2); \tag{40}$$

$$(\mathcal{T}_{\widetilde{N}}(\mathbf{x}_1, \mathbf{x}_2) = \widetilde{T_1 \ldots T_n}_{\widetilde{N}}(\mathbf{x}_1, \mathbf{x}_2) = \widetilde{T_{n_N} \ldots T_{1_N}}(\mathbf{x}_1, \mathbf{x}_2)), \tag{41}$$

is a n-DT (n-DS) which is called $\widetilde{N}$-dual function of $\mathcal{S}$ ($\mathcal{T}$).

Let $\mathcal{T}$ be n-DT. The natural n-DN of $\mathcal{T}$ is the function $\mathcal{N}_{\mathcal{T}} : L_n(U) \to L_n(U)$ given as

$$\mathcal{N}_{\mathcal{T}}(\mathbf{x}) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{z}) = /0/\}. \tag{42}$$

4.5.4 Conjugate operators of n-DT and n-DS on $L_n(U)$

Conjugate operators, in most of the cases, preserve the main properties of the function. For example, if f is an n-dimensional aggregation function, n-DT and Sn-DN, then f^{φ} is also an aggregation function n-dimensional, n-DT and Sn-DN, respectively.

In the following, conjugate operators related to n-DT and n-DS on $L_n(U)$ can be obtained by automorphisms on $L_n(U)$:

Proposition 4.5.4 *(BEDREGAL et al., 2012, Theorem 3.6) Let $\mathcal{T}(\mathcal{S})$ be a n-DT(n-DS) and $\varphi \in Aut(L_n(U)$. Then $\mathcal{T}^{\varphi}(\mathcal{S}^{\varphi}) : (L_n(U))^2 \to L_n(U)$ is an n-DT(n-DS) given as*

$$\mathcal{T}^{\varphi}(\mathbf{x}, \mathbf{y}) = \varphi^{-1}(\mathcal{T}(\varphi(\mathbf{x}), \varphi(\mathbf{y}))); \tag{43}$$

$$\mathcal{S}^{\varphi}(\mathbf{x}, \mathbf{y}) = \varphi^{-1}(\mathcal{S}(\varphi(\mathbf{x}), \varphi(\mathbf{y}))). \tag{44}$$

$\mathcal{T}^{\varphi}(\mathcal{S}^{\varphi})$ is said the conjugate of $\mathcal{T}(\mathcal{S})$.

Proposition 4.5.5 *(BEDREGAL et al., 2011) Let $\widetilde{\psi} \in Aut(L_n(U))$ be a ψ-automorphism and $\widetilde{T_1 \ldots T_n}(\widetilde{S_1 \ldots S_n}) : (L_n(U))^2 \to L_n(U)$ be a representable n-DT (n-DS). The following holds:*

$$\widetilde{T_1 \ldots T_n}^{\widetilde{\psi}}(\mathbf{x}, \mathbf{y}) = \widetilde{T_1^{\psi} \ldots T_n^{\psi}}(\mathbf{x}, \mathbf{y}); \tag{45}$$

$$\widetilde{S_1 \ldots S_n}^{\widetilde{\psi}}(\mathbf{x}, \mathbf{y}) = \widetilde{S_1^{\psi} \ldots S_n^{\psi}}(\mathbf{x}, \mathbf{y}). \tag{46}$$

Proof: Based on (BEDREGAL et al., 2011, Theorem 5), let $\mathbf{x}, \mathbf{y} \in L_n(U)$. Then

$$\begin{aligned}
\widetilde{T_1 \ldots T_n}^{\widetilde{\psi}}(\mathbf{x}, \mathbf{y}) &= \widetilde{\psi}^{-1}(\widetilde{T_1 \ldots T_n}(\widetilde{\psi}(\mathbf{x}), (\widetilde{\psi}(\mathbf{y}))) \\
&= \widetilde{\psi}^{-1}(\widetilde{T_1 \ldots T_n}(\widetilde{\psi}(x_1, \ldots, x_n), \widetilde{\psi}(y_1, \ldots, y_n))) \\
&= \widetilde{\psi}^{-1}(\widetilde{T_1 \ldots T_n}((\psi(x_1), \ldots, \psi(x_n)), (\psi(y_1), \ldots, \psi(y_n)))) \\
&= \widetilde{\psi}^{-1}(T_1(\psi(x_1), \psi(y_1)), \ldots, T_n(\psi(x_n), \psi(y_n))) \\
&= (\psi(T_1(\psi(x_1), \psi(y_1))), \ldots, \psi(T_n(\psi(x_n), \psi(y_n)))) \\
&= T_1^{\psi}(x_1, y_1), \ldots, T_n^{\psi}(x_n, y_n) = \widetilde{T_1^{\psi} \ldots T_n^{\psi}}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Therefore, Proposition 4.5.5 Eq.(45) holds. Analogously for Eq.(46). □

5 FUZZY IMPLICATIONS ON $L_n(U)$

A fuzzy implication can be defined in several ways, such as (BUSTINCE; BURILLO; SORIA, 2003; FODOR; ROUBENS, 1994; FODOR, 1995; SHI et al., 2008; YAGER, 2004b). All these definitions have some aspects in common, therefore , n-Dimensional fuzzy implications can be seen as extensions of interval-valued fuzzy implications (BEDREGAL et al., 2007; ZANOTELLI; REISER; BEDREGAL, 2018a; REISER et al., 2009; BEDREGAL et al., 2010) and interval-valued Atanassov' intuitionistic fuzzy implications (REISER; BEDREGAL, 2013; REISER; BEDREGAL; VISINTIN, 2013). Thus, their properties on U can also be investigated in an n-dimensional sense on $L_n(U)$. This chapter studies n-dimensional fuzzy implications on $(L_n(U), \leq)$, extending the preliminary studies in (CORNELIS; DESCHRIJVER; KERRE, 2004; DESCHRIJVER; CORNELIS; KERRE, 2004).

These studies consider the following steps:

- studies of main properties of fuzzy implications on $L_n(U)$ with emphasis on the conditions that define the continuity of an n-DI;
- study of the conjugation operator defined by the action of automorphisms in the n-dimensional implication class;
- description of expressions defining the $\mathcal{N}$-dual construct for the n-dimensional implication class;
- studies and analysis of the representability of n-DI;
- the analysis of conditions that guarantee the preservation of properties for representable n-dimensional implications.

5.1 Main properties of fuzzy implicators on $L_n(U)$

This section considers studies carried out on n-dimensional fuzzy implications on $L_n(U)$ showing that the main properties of fuzzy implications on U are extended to n-dimensional fuzzy implications on $L_n(U)$. In sequence, this section also verifies the continuity of functions on set of all n-DI ($\mathcal{I}(L_n(U))$) based on the continuity of family $\mathcal{I}(U)$ and discusses the conjugate operators action on n-DI. n-Dimensional fuzzy implications can be seen as extensions of interval-valued fuzzy implications (REISER et al., 2009; BEDREGAL et al., 2007; ZANOTELLI; REISER; BEDREGAL, 2018a; BEDREGAL et al., 2010) and interval-valued Atanassov' intuitionistic fuzzy implications (REISER; BEDREGAL, 2013; REISER; BEDREGAL; VISINTIN, 2013).

Definition 5.1.1 *(ZANOTELLI; REISER; BEDREGAL, 2018a, Definition 7) A function $\mathcal{I} : (L_n(U))^2 \to L_n(U)$ is an n-dimensional fuzzy implicator indicated by n-DI if $\mathcal{I}$ meets the minimal boundary conditions:*
$\mathcal{I}0(a) : \mathcal{I}(/1/, /1/) = \mathcal{I}(/0/, /1/) = \mathcal{I}(/0/, /0/) = /1/$;
$\mathcal{I}0(b) : \mathcal{I}(/1/, /0/) = /0/$.

The corresponding properties of a function $\mathcal{I}$ are given as follows:
$\mathcal{I}1 : \mathbf{x} \leq \mathbf{z} \to \mathcal{I}(\mathbf{x}, \mathbf{y}) \geq \mathcal{I}(\mathbf{z}, \mathbf{y})$;
$\mathcal{I}2 : \mathbf{y} \leq \mathbf{z} \to \mathcal{I}(\mathbf{x}, \mathbf{y}) \leq \mathcal{I}(\mathbf{x}, \mathbf{z})$;
$\mathcal{I}3 : \mathcal{I}(/1/, \mathbf{y}) = \mathbf{y}$;
$\mathcal{I}4 : \mathcal{I}(\mathbf{x}, /1/) = /1/$;
$\mathcal{I}5 : \mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y}, \mathbf{z})) = \mathcal{I}(\mathbf{y}, \mathcal{I}(\mathbf{x}, \mathbf{z}))$;
$\mathcal{I}6 : \mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{x}, \mathbf{y}))$;
$\mathcal{I}7 : \mathcal{I}(\mathbf{x}, \mathcal{N}(\mathbf{x})) = \mathcal{N}(\mathbf{x})$;
$\mathcal{I}8 : \mathcal{I}(\mathbf{x}, \mathbf{y}) \geq \mathbf{y}$;
$\mathcal{I}9 : \mathcal{I}(\mathbf{x}, \mathbf{y}) = \mathcal{I}(\mathcal{N}(\mathbf{y}), \mathcal{N}(\mathbf{x}))$ if $\mathcal{N}$ is Sn-DN;
And, two other conditions are required in $\mathcal{I}$, meaning that
$\mathcal{I}9(a) : \mathcal{I}(\mathbf{x}, \mathcal{N}(\mathbf{y})) = \mathcal{I}(\mathbf{y}, \mathcal{N}(\mathbf{x}))$ (right-contraposition property with respect to $\mathcal{N}$)(R-CP);
$\mathcal{I}9(b) : \mathcal{I}(\mathcal{N}(\mathbf{x}), \mathbf{y}) = \mathcal{I}(\mathcal{N}(\mathbf{y}), \mathbf{x})$ (left-contraposition property with respect to $\mathcal{N}$)(L-CP);
$\mathcal{I}10 : \mathcal{I}(/0/, \mathbf{y}) = /1/$;
$\mathcal{I}11 : \mathcal{I}(\mathbf{x}, \mathbf{y}) = /1/ \Rightarrow \mathbf{x} \leq \mathbf{y}$;
$\mathcal{I}12 : \mathcal{N}(\mathbf{x}) = \mathcal{I}(\mathbf{x}, /0/)$ is a n-DN;
$\mathcal{I}13 : \mathcal{I}(\mathbf{x}, \mathbf{y}) = /1/ \Leftrightarrow \mathbf{x} \leq \mathbf{y}$;
$\mathcal{I}13(a) : \mathcal{I}(\mathbf{x}, \mathbf{y}) = /1/ \Rightarrow \mathbf{x} \preceq \mathbf{y}$;
$\mathcal{I}14 : \mathcal{I}(\mathbf{x}, \mathbf{x}) = /1/$.

Definition 5.1.2 *An n-dimensional fuzzy implicator $\mathcal{I}$ which also satisfies $\mathcal{I}1$ and $\mathcal{I}2$ is called an n-dimensional fuzzy implication (n-DI) or fuzzy implication on $L_n(U)$.*

Since the set of all n-DI (denoted by $\mathcal{I}(L_n(U))$) extends the set of all fuzzy implications (denoted by $I(U)$), their corresponding properties $\mathcal{I}k$ are denoted as Ik for $k \in \mathbb{N}_{14}$. In particular, we can observe that $I13 = I13(a)$ since if $n = 1$ then $\leq$ and $\preceq$ coincide to the usual order on U.

The following Proposition extends results from (BACZYŃSKI; JAYARAM, 2008).

Proposition 5.1.1 *Let $\mathcal{I} : (L_n(U))^2 \to L_n(U)$. If $\mathcal{I}$ satisfies $\mathcal{I}5$ and $\mathcal{I}13$, then it satisfies $\mathcal{I}0(a)$, $\mathcal{I}0(b)$, $\mathcal{I}1$, $\mathcal{I}3$ and $\mathcal{I}14$.*

Proof: Since $/0/ \leq /1/$, by $\mathcal{I}13$, it holds that:
$\mathcal{I}0(a)$: $\mathcal{I}(/0/,/0/) = \mathcal{I}(/0/,/1/) = \mathcal{I}(/1/,/1/) = /1/$.
$\mathcal{I}0(b)$: Straightforward from $\mathcal{I}3$, $\mathcal{I}(/1/,/0/) = /0/$.
$\mathcal{I}1$: If $\mathbf{x}_1 \leq \mathbf{x}_2$, by $\mathcal{I}14$ the following is verified:

$$\begin{aligned}
\mathcal{I}(\mathcal{I}(\mathbf{x}_2,\mathbf{y}),\mathcal{I}(\mathbf{x}_2,\mathbf{y})) = /1/ &\Rightarrow \mathcal{I}(\mathbf{x}_2,\mathcal{I}(\mathcal{I}(\mathbf{x}_2,\mathbf{y}),\mathbf{y})) = /1/ \text{ (by } \mathcal{I}5)\\
&\Rightarrow \mathbf{x}_2 \leq \mathcal{I}(\mathcal{I}(\mathbf{x}_2,\mathbf{y}),\mathbf{y}) \text{ (by } \mathcal{I}13)\\
&\Rightarrow \mathbf{x}_1 \leq \mathcal{I}(\mathcal{I}(\mathbf{x}_2,\mathbf{y}),\mathbf{y}) \Rightarrow \mathcal{I}(\mathbf{x}_1,\mathcal{I}(\mathcal{I}(\mathbf{x}_2,\mathbf{y}),\mathbf{y}) = /1/ \text{ (by } \mathcal{I}13)\\
&\Rightarrow \mathcal{I}(\mathbf{x}_2,\mathbf{y}) \leq \mathcal{I}(\mathbf{x}_1,\mathbf{y}) \text{ (by } \mathcal{I}5).
\end{aligned}$$

$\mathcal{I}3$: By $\mathcal{I}5$ and $\mathcal{I}13$, we consider both conditions:
(i) $\mathcal{I}(\mathbf{y},\mathcal{I}(/1/,\mathbf{y}))=\mathcal{I}(/1/,\mathcal{I}(\mathbf{y},\mathbf{y}))=\mathcal{I}(/1/,/1/)=/1/$. So, this implies that $\mathbf{y}\leq\mathcal{I}(/1/,\mathbf{y})$;
(ii) By $\mathcal{I}14$, $\mathcal{I}(\mathcal{I}(/1/,\mathbf{y}),\mathcal{I}(/1/,\mathbf{y})) = /1/$ meaning that $\mathcal{I}(/1/,\mathcal{I}(\mathcal{I}(/1/,\mathbf{y}),\mathbf{y})) = /1/$. And then, we have that $/1/ \leq \mathcal{I}(\mathcal{I}(/1/,\mathbf{y}),\mathbf{y}) \leq /1/$. So, $\mathcal{I}(\mathcal{I}(/1/,\mathbf{y}),\mathbf{y}) = /1/$. By $\mathcal{I}13$, it also implies that $\mathcal{I}(/1/,\mathbf{y}) \leq \mathbf{y}$.
Therefore, by (i) and (ii), $\mathcal{I}(/1/,\mathbf{y}) = \mathbf{y}$.
$\mathcal{I}14$: By $\mathcal{I}13$, it is immediate that $\mathcal{I}(\mathbf{x},\mathbf{x}) = /1/$.
Therefore, Proposition 5.1.1 holds. □

Analogously, an n-dimensional fuzzy coimplicator (n-DFJ) can be defined as a function $\mathcal{J} : (L_n(U))^2 \to L_n(U)$ meeting the boundary conditions:

$\mathcal{J}0$ (a): $\mathcal{J}(/1/,/1/)= \mathcal{J}(/0/,/1/)=\mathcal{J}(/0/,/0/) = /0/$; $\mathcal{J}0$(b): $\mathcal{J}(/1/,/0/)=/1/$.

Moreover, if $\mathcal{J}$ satisfy $\mathcal{I}1$ and $\mathcal{I}2$ then it is called n-dimensional fuzzy coimplications. And, in the following result, one can achieve the class of n-dimensional fuzzy coimplication based on n-dimensional fuzzy negations and implications.

5.1.1 $\mathcal{N}$-dual fuzzy implicators on $L_n(U)$

The dual function of an n-dimensional implicator can be established if $\mathcal{N}$ be strong n-DN.

Proposition 5.1.2 *Let $\mathcal{I} : (L_n(U))^2 \to L_n(U)$ be an n-dimensional fuzzy implicator and $\mathcal{N} : L_n(U) \to L_n(U)$ be a strong n-DN. The function $\mathcal{I}_\mathcal{N} : (L_n(U))^2 \to L_n(U)$ given as*

$$\mathcal{I}_\mathcal{N}(\mathbf{x},\mathbf{y}) = \mathcal{N}(\mathcal{I}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{y})), \tag{47}$$

and called $\mathcal{N}$-dual of $\mathcal{I}$ is an n-dimensional fuzzy coimplicator. In addition, if $\mathcal{I}$ is an n-dimensional fuzzy implication then $\mathcal{I}_\mathcal{N}$ is a fuzzy coimplication.

Proof: Straightforward from Eq.(32). □

5.1.2 Continuity of n-dimensional fuzzy implications

The condition under which an n-DI verifies the continuity on $\mathcal{I}(L_n(U))$ based on the continuity of family $\mathcal{I}(U^n)$ of fuzzy implications on U^n is considered in the following.

Definition 5.1.3 *Let $\psi : U^n \to L_n(U)$ be the $(U^n, L_n(U))$-permutation expressed by the increase ordering, meaning that $\psi(x_1, \ldots, x_n) = (x_{(1)}, \ldots, x_{(n)})$ such that $\{(1), \ldots, (n)\} = \{1, \ldots, n\}$ and $x_{(i)} \leq x_{(i+1)}, \forall\, i = 1, \ldots, n-1$. A function $\mathcal{F} : L_n(U)^2 \to L_n(U)$ is continuous if the related function $\mathcal{F}^\psi : U^n \times U^n \to U^n$ given by*

$$\mathcal{F}^\psi(\mathbf{x}, \mathbf{y}) = \mathcal{F}(\psi(\mathbf{x}), \psi(\mathbf{y})). \tag{48}$$

is continuous in the usual sense.

Observe that, since $L_n(U) \subset U^n$, then $\mathcal{F}^\psi$ is well defined.

Proposition 5.1.3 *Let $\mathcal{I}$ be a representable n-DI. Then $\mathcal{I}$ is continuous if, and only if $\mathcal{I}_i$ is continuous for each $i = 1, \ldots, n$.*

Proof: ($\Rightarrow$) Let $\psi : U^n \to L_n(U)$ be the $(U^n, L_n(U))$-permutation and $\delta : U^n \to U^n$ the function $\delta(x_1, \ldots, x_n) = (x_n, \ldots, x_1)$. Since, $\widetilde{I_1 \ldots I_n}^\psi$ is continuous and $\widetilde{I_1 \ldots I_n}^\psi = (I_1 \times \cdots \times I_n) \circ ((\delta \circ \psi) \times \psi)$ then each I_i is continuous.

($\Leftarrow$) If I_i is continuous for each $i = 1, \ldots, n$, then $I_1 \times \ldots \times I_n$ also is continuious. Therefore, since $\widetilde{I_1 \ldots I_n}^\psi = (I_1 \times \ldots \times I_n) \circ ((\delta \circ \psi) \times \psi)$ and δ as well as ψ are continuous, $\widetilde{I_1 \ldots I_n}^\psi$ is continuous. Hence, by Definition 5.1.3, $\widetilde{I_1 \ldots I_n}$ is continuous. □

5.1.3 Conjugate-operator $\mathcal{I}^\phi$ on $L_n(U)$

This section discusses the conjugate operators acting on n-DI and showing that the main properties are preserved.

Proposition 5.1.4 *Let $\phi : L_n(U) \to L_n(U)$ be an n-DA and $\mathcal{I} : (L_n(U))^2 \to L_n(U)$. Properties from $\mathcal{I}0$ to $\mathcal{I}14$ are invariant under the conjugate-operator $\mathcal{I}^\phi : (L_n(U))^2 \to L_n(U)$ given by*

$$\mathcal{I}^\phi(\mathbf{x}, \mathbf{y}) = \phi^{-1}\left(\mathcal{I}(\phi(\mathbf{x}), \phi(\mathbf{y}))\right). \tag{49}$$

Proof: Let $\mathcal{I}$ be an n-DI verifying properties from $\mathcal{I}0$ to $\mathcal{I}14$. For $\mathbf{x}, \mathbf{x}_1, \mathbf{x}_2, \mathbf{y} \in L_n(U)$ the following holds:

$\mathcal{I}0$: Next boundary conditions holds straightforward once $\phi(/0/) = \phi^{-1}(/0/) = /0/$ and $\phi(/1/) = \phi^{-1}(/1/) = /1/$ and $\mathcal{I}$ satisfy $\mathcal{I}0$.

$\mathcal{I}1$: Consider $\mathbf{x}_1 \leq \mathbf{x}_2$. By the monotonicity and ϕ^{-1} and because $\mathcal{I}$ satisfy $\mathcal{I}1$, we

obtain that $\mathcal{I}^\phi(\mathbf{x}_1,\mathbf{y}) = \phi^{-1}\left(\mathcal{I}(\phi(\mathbf{x}_1),\phi(\mathbf{y}))\right) \geq \phi^{-1}\left(\mathcal{I}(\phi(\mathbf{x}_2),\phi(\mathbf{y}))\right) = \mathcal{I}^\phi(\mathbf{x}_2,\mathbf{y})$.
$\mathcal{I}2$: Analogous to $\mathcal{I}1$.
$\mathcal{I}3 : \mathcal{I}^\phi(/1/,\mathbf{y}) = \phi^{-1}\left(\mathcal{I}(/1/,\phi(\mathbf{y}))\right) = \phi^{-1}(\phi(\mathbf{y})) = \mathbf{y}$.
$\mathcal{I}4 : \mathcal{I}^\phi(\mathbf{x},/1/) = \phi^{-1}\left(\mathcal{I}(\phi(\mathbf{x}),/1/)\right) = \phi^{-1}(/1/) = /1/$.
$\mathcal{I}5$: Since $\mathcal{I}$ satisfies exchange principle, we have that

$$\begin{aligned}\mathcal{I}^\phi(\mathbf{x},\mathcal{I}^\phi(\mathbf{y},\mathbf{z})) &= \mathcal{I}^\phi(\mathbf{x},\phi^{-1}(\mathcal{I}(\phi(\mathbf{y}),\phi(\mathbf{z})))) = \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}),\mathcal{I}(\phi(\mathbf{y}),\phi(\mathbf{z}))))\\ &= \phi^{-1}(\mathcal{I}(\phi(\mathbf{y}),\mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{z}))))=\mathcal{I}^\phi(\mathbf{y},\phi^{-1}(\mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{z}))))=\mathcal{I}^\phi(\mathbf{y},\mathcal{I}^\phi(\mathbf{x},\mathbf{z})).\end{aligned}$$

$\mathcal{I}6 : \mathcal{I}^\phi(\mathbf{x},\mathbf{y})=\mathcal{I}^\phi(\mathbf{x},\phi^{-1}(\mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{y}))))=\phi^{-1}(\mathcal{I}(\phi(\mathbf{x}),\mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{y})))) = \mathcal{I}^\phi(\mathbf{x},\mathcal{I}^\phi(\mathbf{x},\mathbf{y}))$.

$$\begin{aligned}\mathcal{I}7 : \mathcal{I}^\phi(\mathbf{x},\mathcal{N}(\mathbf{x})) &= \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}),\phi(\mathcal{N}(\mathbf{x})))) = \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}),\mathcal{N}(\phi(\mathbf{x})))\\ &= \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}),\phi^{-1}(\mathcal{N}_{\mathcal{I}}(\phi(\mathbf{x}))) = \mathcal{I}^\phi(\mathbf{x},\mathcal{N}_{\mathcal{I}}^\phi(\mathbf{x})).\end{aligned}$$

$\mathcal{I}8 : \mathcal{I}^\phi(\mathbf{x},\mathbf{y}) = \phi^{-1}(\mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{y}))) \geq \phi^{-1}(\phi(\mathbf{y})) = \mathbf{y}$.

$$\begin{aligned}\mathcal{I}9 : \mathcal{I}^\phi(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{y})) &= \phi^{-1}(\mathcal{I}(\mathcal{N}^\phi(\mathbf{x}),\mathcal{N}^\phi(\mathbf{y}))) = \phi^{-1}(\mathcal{I}(\phi^{-1}(\mathcal{N}(\phi(\mathbf{x})),\mathcal{N}(\phi(\mathbf{y})))))\\ &= \mathcal{I}^\phi(\mathbf{x},\mathbf{y}) \text{ if } \mathcal{I}^\phi \text{ is S}n\text{-DN}.\end{aligned}$$

$\mathcal{I}10 : \mathcal{I}^\phi(/0/,\mathbf{y}) = \phi^{-1}\left(\mathcal{I}(/0/,\phi(\mathbf{y}))\right) = \phi^{-1}(/1/) = /1/$.
$\mathcal{I}11$: Since $\mathcal{I}$ verifies $\mathcal{I}11$, we obtain the next result:

$$\begin{aligned}\mathcal{I}^\phi(\mathbf{x},\mathbf{y}) = /1/ &\Leftrightarrow \phi^{-1}\left(\mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{y})\right) = /1/ \Leftrightarrow (\mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{y}))) = /1/\\ &\Rightarrow \phi(\mathbf{x}) \leq \phi(\mathbf{y}) \Leftrightarrow \mathbf{x} \leq \mathbf{y}.\end{aligned}$$

$\mathcal{I}12 : \mathcal{I}^\phi(\mathbf{x},/0/) = \phi^{-1}\left(\mathcal{I}(\phi(\mathbf{x}),/0/)\right) = \phi^{-1}(\mathcal{N}_{\mathcal{I}}(\phi(\mathbf{x}))) = \mathcal{N}_{\mathcal{I}}{}^\phi(\mathbf{x})$.
$\mathcal{I}13$: Since $\mathcal{I}$ verifies $\mathcal{I}13$, we obtain the next result:

$$\begin{aligned}\mathcal{I}^\phi(\mathbf{x},\mathbf{y}) = /1/ &\Leftrightarrow \phi^{-1}\left(\mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{y}))\right) = /1/ \Leftrightarrow \mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{y})) = /1/\\ &\Leftrightarrow \phi(\mathbf{x}) \leq \phi(\mathbf{y}) \Leftrightarrow \mathbf{x} \leq \mathbf{y}\ .\end{aligned}$$

$\mathcal{I}13(a)$: Since $\mathcal{I}$ verifies $\mathcal{I}13(a)$, we obtain the following equivalence:

$$\begin{aligned}\mathcal{I}^\phi(\mathbf{x},\mathbf{y}) = /1/ &\Leftrightarrow \phi^{-1}\left(\mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{y}))\right) = /1/ \Leftrightarrow \mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{y})) = /1/ \Leftrightarrow \phi(\mathbf{x}) \preceq \phi(\mathbf{y})\\ &\Leftrightarrow \begin{cases} \phi(\mathbf{x}) = \phi(\mathbf{y}) \Leftrightarrow \mathbf{x} = \mathbf{y},\ \text{or}\\ \pi_n(\phi(\mathbf{x})) \leq \pi_1(\phi(\mathbf{y})) \Leftrightarrow \psi(\pi_n(\mathbf{x})) \leq \psi(\pi_1(\mathbf{y})) \Leftrightarrow \pi_n(\mathbf{x}) \leq \psi(\pi_1(\mathbf{y})); \end{cases}\end{aligned}$$

where ψ is the underlying automorphism associated to the n-DA ϕ. So, $\mathcal{I}^\phi(\mathbf{x},\mathbf{y}) = /1/ \Leftrightarrow \mathbf{x} \preceq \mathbf{y}$.
$\mathcal{I}14 : \mathcal{I}^\phi(\mathbf{x},\mathbf{x})=\phi^{-1}\left(\mathcal{I}(\phi(\mathbf{x}),\phi(\mathbf{x}))\right)=\phi^{-1}(/1/)=/1/$.
Concluding, Proposition 5.1.4 holds. □

5.2 Representable fuzzy implications on $L_n(U)$

This section studies the representability property of n-DI showing that the main properties of the fuzzy implications on U are preserved for $L_n(U)$. By verifying a strong $\mathcal{N}$, we can establish the dual function of an n-dimensional implicator. Furthermore, we can preserve the properties of an n-dimensional function using the conjugate.

Proposition 5.2.1 *(ZANOTELLI; REISER; BEDREGAL, 2018a, Proposition 6) Let $I_1, \ldots, I_n : U^2 \to U$ be functions such that $I_1 \leq \ldots \leq I_n$. Then, for all $\mathbf{x}, \mathbf{y} \in L_n(U)$, the function $\widetilde{I_1 \ldots I_n} : (L_n(U))^2 \to L_n(U)$ given by*

$$\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) = (I_1(\pi_n(\mathbf{x}), \pi_1(\mathbf{y})), \ldots, I_n(\pi_1(\mathbf{x}), \pi_n(\mathbf{y}))), \tag{50}$$

is an n-dimensional fuzzy implicator if, and only if, $I_1, \ldots, I_n$ are also fuzzy implicators.

Proof: ($\Rightarrow$) Let $I_1, \ldots, I_n$ be implicators and $I_1 \leq \ldots \leq I_n$.

$$1. \widetilde{I_1 \ldots I_n}(/1/, /1/) = (I_1(1,1), I_2(1,1), \ldots, I_n(1,1)) = (1, 1, \ldots, 1) = /1/;$$
$$2. \widetilde{I_1 \ldots I_n}(/0/, /0/) = (I_1(0,0), I_2(0,0), \ldots, I_n(0,0)) = (0, 0, \ldots, 0) = /0/;$$
$$3. \widetilde{I_1 \ldots I_n}(/0/, /1/) = (I_1(0,1), I_2(0,1), \ldots, I_n(0,1)) = (1, 1, \ldots, 1) = /1/;$$
$$4. \widetilde{I_1 \ldots I_n}(/1/, /0/) = (I_1(1,0), I_2(1,0), \ldots, I_n(1,0)) = (0, 0, \ldots, 0) = /0/.$$

($\Leftarrow$) By applying the projections π_i, for all $1 \leq i \leq n$:

$$\begin{aligned} I_i(x, y) &= \pi_i(\widetilde{I_1, \ldots, I_n}(/x/, /y/)) \\ &= \pi_i(\widetilde{I_1, \ldots, I_n}((x, \ldots, x), (y, \ldots, y)) = \pi_i(I_1(x, y), \ldots, I_n(x, y)). \end{aligned} \tag{51}$$

So, for each $i \in \mathbb{N}_n$ we have that:

$$\widetilde{I_1, \ldots, I_n}(/0/, /0/) = /1/ \to I_i(0,0) = 1;$$
$$\widetilde{I_1, \ldots, I_n}(/0/, /1/) = /1/ \to I_i(0,1) = 1;$$
$$\widetilde{I_1, \ldots, I_n}(/1/, /1/) = /1/ \to I_i(1,1) = 1;$$
$$\widetilde{I_1, \ldots, I_n}(/1/, /0/) = /0/ \to I_i(1,0) = 0.$$

Therefore, Proposition 5.2.1 is verified. □

Based on Proposition 5.2.1, $\mathcal{I}$ is called representable n-DI if there exist fuzzy implicators $I_1 \leq \ldots \leq I_n$ such that $\mathcal{I} = \widetilde{I_1 \ldots I_n}$.

Remark 5.2.1 *When $I_1 = \ldots = I_n = I$, the expression $\widetilde{I_1 \ldots I_n}$ in Eq.(50) is denoted by $\widetilde{I}$.*

Remark 5.2.2 *For $\mathbf{x}, \mathbf{y} \in L_n(U)$, $\widetilde{I_1 \ldots I_n} \in \mathcal{I}(L_n(U))$ and $i \in \mathbb{N}_n$, the following holds:*

(i) $\pi_i(\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y})) = I_i(\pi_{n+1-i}(\mathbf{x}), \pi_i(\mathbf{y}))$;

(ii) $\pi_i(\widetilde{I_1 \ldots I_n}(/x/, /y/)) = I_i(x, y)$;

(iii) $\pi_i(\widetilde{I}(/\mathbf{x}/, /\mathbf{y}/)) = I(x, y)$.

Extending the result from Proposition 5.2.1, next proposition states that $\mathcal{I}$ is a representable n-dimensional fuzzy implication if there exist fuzzy implications $I_1 \leq \ldots \leq I_n$ such that $\mathcal{I} = \widetilde{I_1 \ldots I_n}$, and its converse construction can also be verified.

Proposition 5.2.2 *(ZANOTELLI; REISER; BEDREGAL, 2018a) Let $I_1, \ldots, I_n : U^2 \to U$ be functions such that $I_1 \leq \ldots \leq I_n$. Then, the function $\widetilde{I_1 \ldots I_n} : (L_n(U))^2 \to L_n(U) \in \mathcal{I}(L_n(U))$ if, and only if, $I_i \in I(U)$ for each $i \in \mathbb{N}_n$.*

Proposition 5.2.3 *An n-DI $\widetilde{I_1 \ldots I_n} \in \mathcal{I}(L_n(U))$ is a n-dimensional fuzzy implication if, and only if, $I_1 \ldots I_n$ are also fuzzy implications on U.*

Proof: For $\widetilde{I_1 \ldots I_n}$ given by Eq.(50), it holds that:
$\mathcal{I}1$: ($\Rightarrow$) If $\mathbf{x} \leq \mathbf{z}$ then $(x_1, \ldots, x_n) \leq (z_1, \ldots, z_n)$. Since I_i verifies Property $\mathbf{I}1$, when $x_i \leq z_i$ then $I(x_i, y_i) \geq I(z_i, y_i)$, for $0 \leq i \leq n$, by Property $\mathbf{I}1$ and Eq.(50) meaning that:

$$\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) = (I_1(x_n, y_1), \ldots, I_n(x_1, y_n)) \geq (I_1(z_n, y_1), \ldots, I(z_1, y_n)) = \widetilde{I_1 \ldots I_n}(\mathbf{z}, \mathbf{y}).$$

($\Leftarrow$) If $\mathbf{x} \leq \mathbf{z}$, for $1 \leq i \leq n$, it holds that $x_{n+i-1} \leq z_{n-i+1}$ and therefore $I_n(x_{n+i-1}, y_i) \geq I_n(z_{n+i-1}, y_i)$ So, $\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) \geq \widetilde{I_1 \ldots I_n}(\mathbf{z}, \mathbf{y})$ and $\widetilde{I_1 \ldots I_n}$ verifies $\mathcal{I}1$.
$\mathcal{I}2$: ($\Rightarrow$) If $\mathbf{y} \leq \mathbf{z}$ then $(y_1, \ldots, y_n) \leq (z_1, \ldots, z_n)$. Since I_i verifies Property $\mathbf{I}2$, when $y_i \leq z_i$ then $I(x, y_i) \leq I(x, z_i)$, for $0 \leq i \leq n$. By Property $\mathbf{I}2$ and Eq.(50), we obtain that:

$$\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}_1) = (I_1(x_n, y_1), \ldots, I_n(x_1, y_n)) \leq (I_1(x_n, z_1), \ldots, I(x_1, z_n)) = \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{z}).$$

($\Leftarrow$) If $\mathbf{y} \leq \mathbf{z}$, for $1 \leq i \leq n$ it holds that $y_{n+i-1} \leq z_{n+i-1}$ and therefore $I(x_i, y_{n+i-1}) \leq I(x_i, z_{n+i-1})$. So, $\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) \leq \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{z})$ and $\widetilde{I_1 \ldots I_n}$ verifies $\mathcal{I}2$.
Property $\mathcal{I}2$ can be proved in analogous terms.
Therefore, by above results and Proposition 5.2.1 we can conclude that Proposition 5.2.3 is verified. □

Proposition 5.2.4 *(ZANOTELLI; REISER; BEDREGAL, 2018a, Proposition 7) Let $\widetilde{N}$ be a strong n-DN and $I_1, \ldots, I_n : U^2 \to U$ be functions such that $I_1 \leq \ldots \leq I_n$. The function $\widetilde{I_1 \ldots I_n}_{\widetilde{N}} : (L_n(U))^2 \to L_n(U)$, defined by*

$$\widetilde{I_1 \ldots I_n}_{\widetilde{N}}(\mathbf{x}, \mathbf{y}) = \widetilde{I_{n_N} \ldots I_{1_N}}(\mathbf{x}, \mathbf{y}), \tag{52}$$

where $\widetilde{I_1 \ldots I_n}_{\widetilde{N}}$ is the $\widetilde{N}$-dual function of $\widetilde{I_1 \ldots I_n}$.

Proof: By Eqs.(47) and (50), we have that

$$\begin{aligned}
\widetilde{I_1 \ldots I_n}_{\widetilde{N}}(\mathbf{x}, \mathbf{y}) &= \widetilde{N}(\widetilde{I_1 \ldots I_n}(\widetilde{N}(\mathbf{x}), \widetilde{N}(\mathbf{y}))) \\
&= \widetilde{N}(\widetilde{I_1 \ldots I_n}((N(x_n), \ldots, N(x_1)), (N(y_n), \ldots, N(y_1))) \\
&= \widetilde{N}((I_1(N(x_1), N(y_n)), \ldots, I_n(N(x_n), N(y_1))) \\
&= (N(I_n(N(x_n), N(y_1))), \ldots, N(I_1(N(x_1), N(y_n)))) \\
&= (I_{n_N}(x_n, y_1), \ldots, I_{1_N}(x_1, y_n)) = \widetilde{I_{n_N} \ldots I_{1_N}}(\mathbf{x}, \mathbf{y}), \forall \mathbf{x}, \mathbf{y} \in L_n(U).
\end{aligned}$$

Therefore, Proposition 5.2.4 is verified. □

5.2.1 Preserving properties of representable n-DIs from U

In the following proposition, the main properties of fuzzy implications from U are preserved by the representable n-dimensional fuzzy implications on $L_n(U)$.

Proposition 5.2.5 *Let $I_1, \ldots, I_n : U^2 \to U$ be functions such that $I_1 \leq \ldots \leq I_n$, $i \in \mathbb{N}_n$ and $k \in \{3, 4, 5, 6, 8, 10, 11, 13(a)\}$. An n-DI $\widetilde{I_1 \ldots I_n} : (L_n(U))^2 \to L_n(U)$ verifies the property $\mathcal{I}k$ if, and only if, each $I_i : U^2 \to U$, for $i \in \mathbb{N}_n$, verifies the corresponding property Ik.*

Proof: ($\Leftarrow$) Firstly, let $I_1, \ldots, I_n \in I(U)$ such that $I_1 \leq \ldots \leq I_n$, satisfying property Ik, for $k \in \{3, 4, 5, 6, 8, 10, 11, 13(a)\}$ For $\widetilde{I_1 \ldots I_n} \in \mathcal{I}(L_n(U))$ given by Eq.(50) and $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$, the following holds:

$\mathcal{I}3 : \widetilde{I_1 \ldots I_n}(/1/, \mathbf{y}) = (I_1(1, y_1), \ldots, I_n(1, y_n)) = (y_1, \ldots, y_n) = \mathbf{y}$ (by Eq.(50) and $\mathbf{I}3$).

$\mathcal{I}4 : \widetilde{I_1 \ldots I_n}(\mathbf{x}, /1/) = (I_1(x_n, 1), \ldots, I_n(x_1, 1)) = (1, \ldots, 1) = /1/$ (by Eq.(50) and $\mathbf{I}4$).

$$\begin{aligned}
\mathcal{I}5 : \widetilde{I_1 \ldots I_n}(\mathbf{x}, \widetilde{I_1 \ldots I_n}(\mathbf{y}, \mathbf{z})) &= \widetilde{I_1 \ldots I_n}(\mathbf{x}, (I_1(y_n, z_1), \ldots, I_n(y_1, z_n)) \text{ (by Eq.(50))} \\
&= (I_1(x_n, I_1(y_n, z_1)), \ldots, I_n(x_1, I_n(y_1, z_n))) \text{ (by Eq.(50))} \\
&= (I_1(y_n, I_1(x_n, z_1)), \ldots, I_n(y_1, I_n(x_1, z_n))) \text{ (by } \mathbf{I}5) \\
&= \widetilde{I_1 \ldots I_n}(\mathbf{y}, \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{z})) \text{(by Eq.(50))}.
\end{aligned}$$

$$\begin{aligned}
\mathcal{I}6 : \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) &= (I_1(x_n, y_1), \ldots, I_n(x_1, y_n)) = (I_1(x_n, I_1(x_n, y_1)), \ldots, I_n(x_1, I_n(x_1, y_n))) \\
&= \widetilde{I_1 \ldots I_n}(\mathbf{x}, I_1(x_n, y_1), \ldots, I_n(x_1, y_n)) \\
&= \widetilde{I_1 \ldots I_n}(\mathbf{x}, \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y})) \text{ (by Eq.(50) and } \mathbf{I}6).
\end{aligned}$$

$\mathcal{I}8 : \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) = (I_1(x_n, y_1), \ldots, I_n(x_1, y_n)) \geq (y_1, y_2, \ldots, y_n) = \mathbf{y}$ (by Eq.(50) and $\mathbf{I}8$).

$$\mathcal{I}10 : \widetilde{I_1 \ldots I_n}(/\mathbf{0}/, \mathbf{y}) = (I_1(0, y_1), \ldots, I_n(0, y_n)) = (1, \ldots, 1) = /\mathbf{1}/.$$

$$\mathcal{I}11 : \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) = /1/ \Leftrightarrow (I_1(x_n, y_1), \ldots, I_n(x_1, y_n)) = /1/ \text{ (by Eq.(50))}$$
$$\Leftrightarrow I_1(x_n, y_1) = 1, \ldots, I_n(x_1, y_n) = 1 \Rightarrow x_n \leq y_1 \text{ (by } \mathbf{I}11)$$
$$\Rightarrow x_1 \leq y_1, \ldots, x_n \leq y_n \Rightarrow \mathbf{x} \leq \mathbf{y}.$$

$$\mathcal{I}13(a) : \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) = /1/ \Leftrightarrow (I_1(x_n, y_1), \ldots, I_n(x_1, y_n)) = /1/ \text{ (by Eq.(50))}.$$

Then, by $\mathbf{I}13(a)$ the following holds:

$$\widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}) = /1/ \Leftrightarrow I_1(x_n, y_1) = I_1, \ldots, I_n(x_1, y_n) = 1$$
$$\Leftrightarrow x_1 \leq \ldots \leq x_n \leq y_1 \leq \ldots \leq y_n \Leftrightarrow \mathbf{x} \preceq \mathbf{y}.$$

($\Rightarrow$) Let $\widetilde{I_1, \ldots, I_n} \in \mathcal{I}(L_n(U))$ given by Eq.(50) verifying properties $\mathcal{I}_k$, for $k \in \{3, 4, 5, 6, 8, 10, 11, 13(a)\}$. Based on π_i-projections, for $i \in \mathbb{N}_n$, the following holds for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in U$:

$\mathbf{I}3$: By $\mathcal{I}3, \widetilde{I_1 \ldots I_n}(/1/, /y/) = /y/$ implies $I_i(1, y) = y$.

$\mathbf{I}4$: By $\mathcal{I}4, \widetilde{I_1 \ldots I_n}(/x/, /1/) = /1/$. Then, $I_i(x, 1) = 1$.

$\mathbf{I}5$: By $\mathcal{I}5, \widetilde{I_1 \ldots I_n}(/x/, \widetilde{I_1 \ldots I_n}(/y/, /z/)) = \widetilde{I_1 \ldots I_n}(/y/, \widetilde{I_1 \ldots I_n}(/x/, /z/))$.
So, $I_i(x, I_i(y, z)) = I_i(y, I_i(x, z)), \forall i \in \mathbb{N}_n$.

$\mathbf{I}6$: By $\mathcal{I}6, \widetilde{I_1 \ldots I_n}(/x/, /y/) = \widetilde{I_1 \ldots I_n}(/x/, \widetilde{I_1 \ldots I_n}(/x, /y/))$.
Then we obtain that $(I_i(x, y) = I_i(x, (I_i(x, y))$.

$\mathbf{I}8$: By $\mathcal{I}8, \widetilde{I_1 \ldots I_n}(/x/, /y/) \geq /y/$. Then $I_i(x, y) \geq y$.

$\mathbf{I}10$: By $\mathcal{I}10, \widetilde{I_1 \ldots I_n}(/0/, /y/) = /1/$ Then $I_i(0, y) = 1$.

$\mathbf{I}11$: By $\mathcal{I}11, \widetilde{I_1 \ldots I_n}(/x/, /y/) = /1/ \Rightarrow /x/ \leq /y/$. Then, we have that $I_i(x, y) = 1 \Rightarrow x \leq y$.

$\mathbf{I}13(a)$: By $\mathcal{I}13(a), \widetilde{I_1 \ldots I_n}(/x/, /y/) = /1/ \Leftrightarrow /x/ \preceq /y/$.Then, $I_i(x, y) = 1 \Leftrightarrow x \leq y$.

Therefore, Proposition 5.2.5 is verified. □

Proposition 5.2.6 *Let $\widetilde{N_1 \ldots N_n}$ be an n-DN and $\widetilde{I_1 \ldots I_n} \in \mathcal{I}(L_n(U))$. A pair $(\widetilde{N_1 \ldots N_n}, \widetilde{I_1 \ldots I_n})$ verifies $\mathcal{I}7$ and $\mathcal{I}12$ if, and only if, for each $i = 1, \ldots, n$, (N_i, I_i) verifies corresponding properties $\mathbf{I}7$ and $\mathbf{I}12$.*

Proof: ($\Rightarrow$) First, let (N_i, I_i) be a pair of an fuzzy implication and fuzzy negation such that I_i verifies $\mathbf{I}7$ and $\mathbf{I}12$, for each $i = 1, \ldots, n$.

$$\mathcal{I}7 : \widetilde{I_1 \ldots I_n}(\mathbf{x}, \widetilde{N_1 \ldots N_n}(\mathbf{x})) = (I_1(x_n, N_1(x_n)), \ldots, I_n(x_1, N_n(x_1))) \text{ (by Eq.(50))}$$
$$= (N_1(x_n), \ldots, N_n(x_1)) = \mathcal{N}(\mathbf{x}) \text{ (by Eq.(30) and } \mathbf{I}7) .$$

Then, $(\widetilde{N_1 \ldots N_n}, \widetilde{I_1 \ldots I_n})$ verifies $\mathcal{I}7$. Additionally, the following holds:

$$\begin{aligned}\mathcal{I}12 : \widetilde{I_1 \ldots I_n}(\mathbf{x}, /0/) &= (I_1(x_n, 0), \ldots, I_n(x_1, 0))\text{(by Eq.(50))}\\ &= (N_{I_1}(x_n), \ldots, N_{I_n}(x_1)) = \widetilde{N_{I_1} \ldots N_{I_n}}(\mathbf{x}) \text{ (by Eq.(30) and } \mathbf{I}12).\end{aligned}$$

Since $N_{I_i} \leq N_{I_j}$ when $i \leq j$, meaning that $\widetilde{I_1 \ldots I_n}(\mathbf{x}, /0/) = \widetilde{N_{I_1} \ldots N_{I_n}}$ is an n-DN.
$(\Leftarrow)$ Conversely, if $\widetilde{I_1 \ldots I_n}(\mathbf{x}, \widetilde{N_1 \ldots N_n}(\mathbf{x})) = \widetilde{N_1 \ldots N_n}(\mathbf{x})$, by Property $\mathcal{I}7$, then we have that $(I_i(x_{n-i+1}, N_i(x_{n-i+1})) = N_i(x_{n-i+1})$. Moreover, when a pair $(\widetilde{N_1 \ldots N_n}, \widetilde{I_1 \ldots I_n})$ verifies $\mathcal{I}12$, by projection π_i, for each $i = 1, \ldots$, we have that $I_i(x_{n-i+1}, 0) = N_i(x_{n-i+1})$, meaning that (N_i, I_i) verifies the corresponding property $\mathbf{I}12$. Therefore, Proposition 5.2.6 is verified. □

Proposition 5.2.7 *A pair* $\left(\widetilde{I_1 \ldots I_n}, \widetilde{N}\right)$ *verifies property* $\mathcal{I}9$ *if, and only if, for each* $i = 1, \ldots, n$*, the pairs* (I_i, N) *verify* $\mathbf{I}9$*.*

Proof: $(\Rightarrow)$ First, for each $i = 1, \ldots, n$, according to (BEDREGAL et al., 2010, Remark 18), if a pair (I_i, N) verifies $\mathbf{I}9$, it holds that:

$$\begin{aligned}\widetilde{I_1 \ldots I_n}(\widetilde{N}(\mathbf{y}), \widetilde{N}(\mathbf{x})) &= \widetilde{I_1 \ldots I_n}((N(y_n), \ldots, N(y_1)), (N(x_n), \ldots, N(x_1))) \text{ (by Eq.(30))}\\ &= (I_1(N(y_1), N(x_n), \ldots I_n(N(y_n), N(x_1)) \text{ (by Eq.(50))}\\ &= (I_1(x_n, y_1), \ldots, I_n(x_1, y_n)) = \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y}).\end{aligned}$$

$(\Leftarrow)$ Now, when $\widetilde{I_1 \ldots I_n}$ verifies $\mathcal{I}9$, meaning that $\widetilde{I_1 \ldots I_n}(\widetilde{N}(\mathbf{y}), \widetilde{N}(\mathbf{x})) = \widetilde{I_1 \ldots I_n}(\mathbf{x}, \mathbf{y})$, by projections π_i, for each $i = 1, \ldots, n$ the following holds $I_i(N(y_{n-i+1}), N(x_i)$. Therefore, Proposition 5.2.7 is verified. □

Proposition 5.2.8 *Let* $\widetilde{I_1 \ldots I_n} : (L_n(U))^2 \to L_n(U)$ *such that* $I_1 \leq \ldots \leq I_n$ *are fuzzy implicators on* U*. Then* $\widetilde{I_1 \ldots I_n}$ *does not verify the properties* $\mathcal{I}13$ *and* $\mathcal{I}14$*.*

Proof: Suppose that each $I_i{:}U^2 \to U$, for $i{\in}\mathbb{N}_n$, verifies both properties, $I13$ and $I14$. It is immediate that $\widetilde{I_1, \ldots, I_n}$ does not verifies $\mathcal{I}14$, by taking $\mathbf{x}{=}(0, ..., 0, 1)$, we have that

$$\widetilde{I_1, \ldots, I_n}(\mathbf{x}, \mathbf{x}) = (I_1(1, 0), I_2(0, 1), ..., I_n(0, 1)) = (0, 1, ..., 1) \neq /1/.$$

Moreover, it does not verify $\mathcal{I}13$. Indeed, suppose that $I_1(1, b) = a \neq 1$. Then, by Eq.(50), for some $a, b \in U$:

$$\widetilde{I_1, \ldots, I_n}((b/2, 1, \ldots, 1), (b, 1, \ldots, 1)){=}(I_1(1, b), I_2(1, 1), ..., I_n(b/2, 1)) = (a, 1, ..., 1) \neq /1/.$$

So, $\mathbf{x} = (b/2, 1, \ldots, 1) \leq (b, 1, \ldots, 1) = \mathbf{y}$ but $\widetilde{I_1, \ldots, I_n}(\mathbf{x}, \mathbf{y}) \neq /1/$. □

Example 5.2.1 *Based on Eq.(7) reported in Example 3.3.1, an example of a representable n-dimensional fuzzy implications is presented:*

Let, $\widetilde{I_{KD}, I_{RC}, I_{LK}, I_{WB}}$ *be a representable n-DI on $L_n(U)$. When* $\mathbf{x} = (0.0, 0.1, 0.5, 0.8)$ *and* $\mathbf{y} = (0.2, 0.6, 0.9, 1.0)$*, it holds that* $\mathcal{I}(\mathbf{x}, \mathbf{y}) = \widetilde{I_{KD}, I_{RC}, I_{LK}, I_{WB}}(\mathbf{x}, \mathbf{y}) = (0.2, 0.8, 1, 1)$*. Moreover, according to Remark 5.2.1, we have that:*

$$\widetilde{I_{KD}}(\mathbf{x}, \mathbf{y}) = (I_{KD}(0.8, 0.2), I_{KD}(0.5, 0.6), I_{KD}(0.1, 0.9), I_{KD}(0, 1)) = (0.2, 0.6, 0.9, 1.0);$$
$$\widetilde{I_{RC}}(\mathbf{x}, \mathbf{y}) = (I_{RC}(0.8, 0.2), I_{RC}(0.5, 0.6), I_{RC}(0.1, 0.9), I_{RC}(0, 1)) = (0.36, 0.8, 0.99, 1.0);$$
$$\widetilde{I_{LK}}(\mathbf{x}, \mathbf{y}) = (I_{LK}(0.8, 0.2), I_{LK}(0.5, 0.6), I_{LK}(0.1, 0.9), I_{LK}(0, 1)) = (0.4, 1.0, 1.0, 1.0);$$
$$\widetilde{I_{WB}}(\mathbf{x}, \mathbf{y}) = (I_{WB}(0.8, 0.2), I_{WB}(0.5, 0.6), I_{WB}(0.1, 0.9), I_{WB}(0, 1)) = (1.0, 1.0, 1.0, 1.0).$$

Based on previous studies on intuitive notion and main properties of n-DI, we introduce in next theorem the notion of Moore-continuous functions for representable n-DI.

Theorem 5.2.1 *An n-DI* $\widetilde{I_1 \dots I_n} : (L_n(U))^2 \to L_n(U)$ *is Moore-continuous if, and only if, every I_i, $i \in \mathbb{N}_n$, is continuous.*

Proof: $(\Rightarrow)$ Let $(x_1, y_1), (x_2, y_2) \in U^2$, $i \in \mathbb{N}_n$ and $\varepsilon > 0$. Since $\widetilde{I_1 \dots I_n}$ is a Moore-continuous function on $L_n(U)$ and $(/x_1/, /y_1/), (/x_2/, /y_2/) \in (L_n(U))^2$. So, there exists $\delta > 0$ such that we obtain the following:

$$d_M^{n,2}((/x_1/, /y_1/), (/x_2/, /y_2/)) < \delta$$

implies that

$$d_M^n(\widetilde{I_1 \dots I_n}(/x_1/, /y_1/), \widetilde{I_1 \dots I_n}(/x_2/, /y_2/)) < \varepsilon.$$

Thus, if $\max(|x_1 - x_2|, |y_1 - y_2|) < \delta$ then, by Eq.(22), we have that $d_M^{n,2}((/x_1/, /y_1/), (/x_2/, /y_2/)) < \delta$. And, one can easily observe that $d_M^n(\widetilde{I_1 \dots I_n}(/x_1/, /y_1/), \widetilde{I_1 \dots I_n}(/x_2/, /y_2/)) < \varepsilon$. So, it means that the following holds

$$d_M^n((I_1(x_1, y_1), \dots, I_n(x_1, y_1)), (I_1(x_2, y_2), \dots, I_n(x_2, y_2))) < \varepsilon.$$

So, by Eq.(21), $|I_i(x_1, y_1) - I_i(x_2, y_2)| < \varepsilon$. Therefore, I_i is continuous.

$(\Leftarrow)$ Let $\varepsilon > 0$ and $(\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2) \in (L_n(U))^2$. By continuity of I_i, for each $i \in \mathbb{N}_n$ and $(\pi_i(\mathbf{x}_1), \pi_i(\mathbf{y}_1)) \in U^2$, there exists $\delta_i > 0$ such that $\forall (\pi_i(\mathbf{x}_2), \pi_i(\mathbf{y}_2)) \in U^2$, whenever $\max(|\pi_i(\mathbf{x}_1) - \pi_i(\mathbf{x}_2)|, |\pi_i(\mathbf{y}_1) - \pi_i(\mathbf{y}_2)|) < \delta_i$ then $|I_i(\pi_i(\mathbf{x}_1), \pi_i(\mathbf{y}_1)) - I_i(\pi_i(\mathbf{x}_2), \pi_i(\mathbf{y}_2))| <$

ε. Thus, considering $\delta = \min\{\delta_i : i \in \mathbb{N}_n\}$, we have that

$$
\begin{aligned}
& d_M^{n,2}((\mathbf{x}_1, \mathbf{y}_1), (\mathbf{x}_2, \mathbf{y}_2)) < \delta \Rightarrow \max(d_M^n(\mathbf{x}_1, \mathbf{x}_2), d_M^n(\mathbf{y}_1, \mathbf{y}_2)) < \delta \\
& \Rightarrow \max(|\pi_1(\mathbf{x}_1) - \pi_1(\mathbf{x}_2)|, \ldots, |\pi_n(\mathbf{x}_1) - \pi_n(\mathbf{x}_2)|, \\
& \quad |\pi_1(\mathbf{y}_1) - \pi_1(\mathbf{y}_2)|, \ldots, |\pi_n(\mathbf{y}_1) - \pi_n(\mathbf{y}_2)|) < \delta \\
& \Rightarrow \max(|\pi_{n-i+1}(\mathbf{x}_1) - \pi_{n-i+1}(\mathbf{x}_2)|, |\pi_i(\mathbf{y}_1) - \pi_i(\mathbf{y}_2)|) < \delta_i, \text{ for all } i \in \mathbb{N}_n \\
& \Rightarrow |I_i(\pi_{n-i+1}(\mathbf{x}_1), \pi_i(\mathbf{y}_1)) - I_i(\pi_{n-i+1}(\mathbf{x}_2), \pi_i(\mathbf{y}_2))| < \varepsilon \text{ for all } i \in \mathbb{N}_n \\
& \Rightarrow \max(|I_1(\pi_n(\mathbf{x}_1), \pi_1(\mathbf{y}_1)) - I_1(\pi_n(\mathbf{x}_2), \pi_1(\mathbf{y}_2)|, \ldots, \\
& \quad |I_n(\pi_1(\mathbf{x}_1), \pi_n(\mathbf{y}_1)) - I_n(\pi_1(\mathbf{x}_2), \pi_n(\mathbf{y}_2)|) < \varepsilon \\
& \Rightarrow \max(|\pi_1(\widetilde{I_1, \ldots, I_n}(\mathbf{x}_1, \mathbf{y}_1)) - \pi_1(\widetilde{I_1, \ldots, I_n}(\mathbf{x}_2, \mathbf{y}_2))|, \ldots, \\
& \quad |\pi_n(\widetilde{I_1, \ldots, I_n}(\mathbf{x}_1, \mathbf{y}_1)) - \pi_n(\widetilde{I_1, \ldots, I_n}(\mathbf{x}_2, \mathbf{y}_2))|) < \varepsilon \\
& \Rightarrow d_M^n(\widetilde{I_1, \ldots, I_n}(\mathbf{x}_1, \mathbf{y}_1), \widetilde{I_1, \ldots, I_n}(\mathbf{x}_2, \mathbf{y}_2)) < \varepsilon.
\end{aligned}
$$

Therefore, $\widetilde{I_1 \ldots I_n}$ is a $(d_M^{n,2}, d_M^n)$-continuous function on $L_n(U) \times L_n(U)$ and Theorem 5.2.1 is verified. □

5.2.2 Conjugate fuzzy implicators on $L_n(U)$

The conjugated function of an n-DI, can be established from an automorphism.

Proposition 5.2.9 *Let $\widetilde{\psi} \in Aut(L_n(U))$ be a ψ-representable automorphism and $\widetilde{I_1 \ldots I_n} : (L_n(U))^2 \to L_n(U)$ be a representable n-DI. Then*

$$
\widetilde{I_1 \ldots I_n}^{\widetilde{\psi}}(\mathbf{x}, \mathbf{y}) = \widetilde{I_1^{\psi} \ldots I_n^{\psi}}(\mathbf{x}, \mathbf{y}). \tag{53}
$$

Proof: For all $\mathbf{x}, \mathbf{y} \in L_n(U)$, we have that

$$
\begin{aligned}
\widetilde{I_1 \ldots I_n}^{\widetilde{\psi}}(\mathbf{x}, \mathbf{y}) &= \widetilde{\psi}^{-1}(\widetilde{I_1 \ldots I_n}(\widetilde{\psi}(\mathbf{x}), (\widetilde{\psi}(\mathbf{y}))) \\
&= \widetilde{\psi}^{-1}(\widetilde{I_1 \ldots I_n}(\psi(x_1, \ldots, x_n), \psi(y_1, \ldots, y_n))) \\
&= \widetilde{\psi}^{-1}(\widetilde{I_1 \ldots I_n}((\psi(x_1), \ldots, \psi(x_n)), (\psi(y_1), \ldots, \psi(y_n)))) \\
&= \widetilde{\psi}^{-1}(I_1(\psi(x_n), \psi(y_1)), \ldots, I_n(\psi(x_1), \psi(y_n))) \\
&= \widetilde{\psi^{-1}}(I_1(\psi(x_n), \psi(y_1)), ..., I_n(\psi(x_1), \psi(y_n))) \\
&= (\psi^{-1}(I_1(\psi(x_n), \psi(y_1))), \ldots, \psi^{-1}(I_n(\psi(x_1), \psi(y_n)))) \\
&= I_1^{\psi}(x_n, y_1), \ldots, I_n^{\psi}(x_1, y_n) = \widetilde{I_1^{\psi} \ldots I_n^{\psi}}(\mathbf{x}, \mathbf{y}).
\end{aligned}
$$

Therefore, Proposition 5.2.9 is verified. □

5.2.3 Conjugation of $\mathcal{N}$-dual fuzzy implicators on $L_n(U)$

For a strong N-representable n-DN and a ψ-representable conjugate operator, the composition $\widetilde{N} \circ \widetilde{\psi}$ is symmetric on $L_n(U)$ whenever the composition $\psi \circ N$ is symmetric in U. It means that such composition is invariant with respect to the n-dimensional approach.

Proposition 5.2.10 *Let $\widetilde{\psi} \in Aut(L_n(U))$ be a ψ-representable automorphism, $\widetilde{I_1 \ldots I_n} : (L_n(U))^2 \to L_n(U)$ be a representable n-DI and $\widetilde{N}$ be strong n-DN. Then the following hols:*

$$\left(\widetilde{I_1 \ldots I_n}_{\widetilde{N}}\right)^{\widetilde{\psi}}(\mathbf{x}, \mathbf{y}) = \left(\widetilde{I_1 \ldots I_n}_{\widetilde{\psi}}\right)^{\widetilde{N}}(\mathbf{x}, \mathbf{y}). \tag{54}$$

Proof: Let $\widetilde{\psi}$ be an ψ-representable automorphism and $\widetilde{I_1 \ldots I_n}$ be a representable n-DI and $\widetilde{N}$ be a strong n-DN. For all $\mathbf{x}, \mathbf{y} \in L_n(U)$, and based on results from Eq.(52) and Eq.(53) the following holds:

$$\begin{aligned}
\left(\widetilde{I_1 \ldots I_n}_{\widetilde{N}}\right)^{\widetilde{\psi}}(\mathbf{x}, \mathbf{y}) &= \widetilde{\psi}^{-1}(\widetilde{I_1 \ldots I_n})_{\widetilde{N}}(\widetilde{\psi}(\mathbf{x}), \widetilde{\psi}(\mathbf{y})) \\
&= \widetilde{\psi}^{-1}(\widetilde{I_1 \ldots I_n})_{\widetilde{N}}((\psi(x_1), \ldots, \psi(x_n)), (\psi(y_1), \ldots, \psi(y_n))) \\
&= \widetilde{\psi}^{-1} \circ \widetilde{N}(\widetilde{I_1 \ldots I_n})((N(\psi(y_n), \ldots, N(\psi(y_1))), (N(\psi(x_n), \ldots, N(\psi(x_1)))) \\
&= \widetilde{\psi}^{-1} \circ \widetilde{N}(I_1(N(\psi(x_1)), N(\psi(y_n))), \ldots, I_n(N(\psi(x_n)), N(\psi(y_1)))) \\
&= \widetilde{\psi^{-1}}(N(I_n(N(\psi(x_n)), N(\psi(y_1)))), \ldots, N(I_1(N(\psi(x_1)), N(\psi(y_n))))) \\
&= (\psi^{-1}(N(I_n(N(\psi(x_n)), N(\psi(y_1))))), \ldots, \psi^{-1}(N(I_1(N(\psi(x_1)), N(\psi(y_n)))))) \\
&= (\psi^{-1}(I_{n_N}(\psi(x_n), \psi(y_1))), \ldots, \psi^{-1}(I_{1_N}(\psi(x_1)), \psi(y_n)))) \\
&= ((I_{n_N})^{\psi}(x_n, y_1), \ldots, (I_{1_N})^{\psi}(x_1, y_n)) \\
&= ((I_n^{\psi})_N(x_n, y_1), \ldots, (I_1^{\psi})_N(x_1, y_n)) \\
&= (\widetilde{I_1^{\psi} \ldots I_n^{\psi}})_{\widetilde{N}}(\mathbf{x}, \mathbf{y}) = (\widetilde{I_1 \ldots I_n}^{\psi})_{\widetilde{N}}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Therefore, Proposition 5.2.10 is verified. □

5.3 Extending main classes of fuzzy implications

The literature presents many classes of fuzzy implications (see (DUBOIS; PRADE, 2000; BACZYŃSKI; JAYARAM, 2008a; BACZYŃSKI; JAYARAM, 2009; BACZYŃSKI; JAYARAM, 2008)). In this study, three classes are considered, (S, N)-implication, QL-implication and R-implication. In the following, we will describe a formalization for the class of (S,N)-implications and QL-implications in the lattice $L_n(U)$, leaving the R-implication on $L_n(U)$ for the next chapter.

5.3.1 $(\mathcal{S},\mathcal{N})$-implications on $L_n(U)$

The following is a study of the properties that satisfy the $(\mathcal{S},\mathcal{N})$-implications in the $L_n(U)$ approach.

Proposition 5.3.1 *Let $\mathcal{S}$ be an n-DS and $\mathcal{N}$ be a n-DN. The function $\mathcal{I}_{\mathcal{S},\mathcal{N}} : (L_n(U))^2 \to L_n(U)$ given by*

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x},\mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathbf{x}),\mathbf{y}), \tag{55}$$

is an n-DI called as n-dimensional $(\mathcal{S},\mathcal{N})$-implication.

Proof: Let $\mathcal{S}$ be an n-DS and $\mathcal{N}$ be an n-DN. The following holds:

$\mathcal{I}0$: The boundary conditions $\mathcal{I}0(a)$ and $\mathcal{I}0(b)$ are verified:
$\mathcal{I}_{\mathcal{S},\mathcal{N}}(/1/,/1/) = \mathcal{S}(/0/,/1/) = /1/$; $\mathcal{I}_{\mathcal{S},\mathcal{N}}(/0/,/1/) = \mathcal{S}(/1/,/0/) = /1/$;
$\mathcal{I}_{\mathcal{S},\mathcal{N}}(/0/,/0/) = \mathcal{S}(/1/,/0/) = /1/$; $\mathcal{I}_{\mathcal{S},\mathcal{N}}(/1/,/0/) = \mathcal{S}(/0/,/0/) = /0/$.

$\mathcal{I}1$**:** $\mathbf{x} \le \mathbf{z} \Rightarrow \mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x},\mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathbf{x}),\mathbf{y}) \ge \mathcal{S}(\mathcal{N}(\mathbf{z}),\mathbf{y}) = \mathcal{I}(\mathbf{z},\mathbf{y})$, based on both properties, the monotonicity of $\mathcal{S}$ and the monotonicity of $\mathcal{N}$.

$\mathcal{I}2$**:** Analogously, $\mathbf{y} \le \mathbf{z} \Rightarrow \mathcal{I}(\mathbf{x},\mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathbf{x}),\mathbf{y}) \le \mathcal{S}(\mathcal{N}(\mathbf{x}),\mathbf{z}) = \mathcal{I}(\mathbf{x},\mathbf{z})$, based on the monotonicity of $\mathcal{S}$.

Therefore, $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ satisfies the Properties $\mathcal{I}1$, $\mathcal{I}2$ and Proposition 5.3.1 is verified. □

Remark 5.3.1 *The underlying n-DS and n-DN of an n-dimensional $(\mathcal{S},\mathcal{N})$-implication $\mathcal{I}$ are called of generators pair. Let $\mathcal{N}$ be an involutive function. $\mathcal{I}_{S,N}$ is denoted by $\mathcal{I}_S$ called $\mathcal{S}$-implication.*

Proposition 5.3.2 *If $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is an n-dimensional $(\mathcal{S},\mathcal{N})$-implication then*

(i) $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ verifies $\mathcal{I}3$ and $\mathcal{I}5$;

(ii) $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ verifies $\mathcal{I}12$: $\mathcal{N}_{\mathcal{I}_{\mathcal{S},\mathcal{N}}} = \mathcal{N}$;

(iii) $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ verifies the right-contraposition property: $\mathcal{I}9a$: $\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x},\mathbf{y}) = I(\mathbf{y},\mathcal{N}(\mathbf{x}))$ with respect to $\mathcal{N}^{-1}$;

(iv) If $\mathcal{N}$ is strict then $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ verifies the left-contraposition property: $\mathcal{I}9b$: $\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x},\mathbf{y}) = I(\mathcal{N}(\mathbf{y}),\mathbf{x})$ with respect to $\mathcal{N}$;

(v) If $\mathcal{N}$ is strong then $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ verifies the contraposition property $\mathcal{I}9$.

Proof: Let $\mathcal{S}$ be an n-DS and $\mathcal{N}$ be an n-DN. For all $\mathbf{x},\mathbf{y},\mathbf{z}$, the following holds:

$\mathcal{I}3$: $\mathcal{I}_{\mathcal{S},\mathcal{N}}(/1/,\mathbf{y}) = \mathcal{S}(/0/,\mathbf{y}) = \mathbf{y}$.

$\mathcal{I}5$: Since $\mathcal{S}$ verifies the $\mathcal{S}2$ and $\mathcal{S}3$ properties, the following holds:

$$\begin{aligned}\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{S}(\mathcal{N}(\mathbf{y}),\mathbf{z}))) &= \mathcal{S}(\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{N}(\mathbf{y}),\mathbf{z}))) = \mathcal{S}(\mathcal{S}(\mathcal{N}(\mathbf{y}),\mathcal{N}(\mathbf{x}),\mathbf{z})))\\ &= \mathcal{S}(\mathcal{N}(\mathbf{y}),\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathbf{z}))).\end{aligned}$$

Therefore, $\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x},(\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{y},\mathbf{z}) = \mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{y},(\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x},\mathbf{z})$.

$\mathcal{I}12$: $\mathcal{N}_{\mathcal{I}_{\mathcal{S},\mathcal{N}}}(\mathbf{x}) = \mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x},/0/) = \mathcal{S}(\mathcal{N}(\mathbf{x}),/0/) = \mathcal{N}(\mathbf{x})$.

$\mathcal{I}9$**a** :Since $\mathcal{N}$ is strict, there exists $\mathbf{y}' \in L_n(U)$ such that $\mathcal{N}^{-1}(\mathbf{y}') = \mathbf{x}$.Then,
$\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x},\mathbf{y})=\mathcal{S}(\mathcal{N}(\mathbf{x}),\mathbf{y})=\mathcal{S}(\mathbf{y},\mathcal{N}(\mathbf{x}))=\mathcal{S}(\mathbf{y},\mathcal{N}(\mathcal{N}(\mathbf{x}))=\mathcal{S}(\mathbf{y},\mathbf{x})=\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{y},\mathcal{N}(\mathbf{x}))$.

$\mathcal{I}9$**b** : If $\mathcal{N}$ is strict then exist $\mathcal{N},\mathcal{N}^{-1} : L_n(U) \to L_n(U)$ such that $\mathcal{N}\circ\mathcal{N}^{-1}(\mathbf{x}) = \mathbf{x}$. And, it holds that:

$$\begin{aligned}\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathcal{N}^{-1}(\mathbf{x}),\mathbf{y}) &= \mathcal{S}(\mathcal{N}(\mathcal{N}^{-1}(\mathbf{x})),\mathbf{y}) = \mathcal{S}(\mathbf{x},\mathbf{y}) = \mathcal{S}(\mathbf{y},\mathbf{x}) \text{ by Properties } \mathcal{N}3(A) \text{ and } \mathcal{S}3\\ &= \mathcal{S}(\mathcal{N}(\mathcal{N}^{-1}(\mathbf{y}),\mathbf{x}))) = \mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathcal{N}^{-1}(\mathbf{y}),\mathbf{x}).\end{aligned}$$

$\mathcal{I}9$: $\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathcal{N}(\mathbf{y}),\mathcal{N}(\mathbf{x})) = \mathcal{S}(\mathcal{N}^2(\mathbf{y}),\mathcal{N}(\mathbf{x})) = \mathcal{S}(\mathbf{y},\mathcal{N}(\mathbf{x})) = \mathcal{S}(\mathcal{N}(\mathbf{x}),\mathbf{y}) = \mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x},\mathbf{y})$.

Therefore, Proposition 5.3.2 is verified. □

Proposition 5.3.3 *Let $\mathcal{I} : (L_n(U))^2 \to L_n(U)$ be an n-DI verifying $\mathcal{I}3,\mathcal{I}4,\mathcal{I}10$ and $\mathcal{N}$ be a n-DN. If $\mathcal{S}_{\mathcal{I},\mathcal{N}} : (L_n(U))^2 \to L_n(U)$ is a given function as follows:*

$$\mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},\mathbf{y}) = \mathcal{I}(\mathcal{N}(\mathbf{x}),\mathbf{y}),\ \ \mathbf{x},\mathbf{y} \in L_n(U).$$

Then the following holds:

(i) $\mathcal{S}_{\mathcal{I},\mathcal{N}}(/1/,\mathbf{x}) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},/1/)$*;*

(ii) $\mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x},\mathbf{y})$ *is increasing in both variables;*

(iii) $\mathcal{S}$ *is commutative iff* $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ *satisfies L-CP*(N)*;*

In addition, if $\mathcal{I}$ *satisfies L-CP*(N)*, then*

(iv) $\mathcal{S}_{\mathcal{I},\mathcal{N}}$ *satisfies* $S4$ *iff* $\mathcal{I}$ *satisfies* $\mathcal{I}3$*;*

(v) $\mathcal{S}_{\mathcal{I},\mathcal{N}}$ *is associative iff* $\mathcal{I}$ *verifies* $\mathcal{I}5$*.*

Proof: For all $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$, we obtain the following results:

$(i)\mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}, /1/) = \mathcal{I}(\mathcal{N}(\mathbf{x}), /1/) = /1/$ by $\mathcal{I}4$;
$\mathcal{S}_{\mathcal{I},\mathcal{N}}(/1/, \mathbf{x}) = \mathcal{I}(\mathcal{N}(/1/), \mathbf{x}) = /1/$ by $\mathcal{I}10$.
$(ii)\, \mathbf{x}_1 \leq \mathbf{x}_2 \rightarrow \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}_1, \mathbf{y}) = \mathcal{I}(\mathcal{N}(\mathbf{x}_1), \mathbf{y}) \leq \mathcal{I}(\mathcal{N}(\mathbf{x}_2), \mathbf{y}) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}_2, \mathbf{y})$;
$\mathbf{y}_1 \leq \mathbf{y}_2 \rightarrow \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}, \mathbf{y}_1) = \mathcal{I}(\mathcal{N}(\mathbf{x}), \mathbf{y}_1) \leq \mathcal{I}(\mathcal{N}(\mathbf{x}), \mathbf{y}_2) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}, \mathbf{y}_2)$.
$(iii)(\Leftarrow)\mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}, \mathbf{y}) = \mathcal{I}(\mathcal{N}(\mathbf{x}), \mathbf{y}) = \mathcal{I}(N(\mathbf{y}), \mathbf{x}) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{y}, \mathbf{x})$ by $\mathcal{I}$9b;
$(\Rightarrow)\mathcal{I}(N(\mathbf{x}), \mathbf{y}) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}, \mathbf{y}) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{y}, \mathbf{x}) = \mathcal{I}(N(\mathbf{y}), \mathbf{x})$ by $\mathcal{S}4$.
$(iv)\mathcal{S}_{\mathcal{I},\mathcal{N}}(/0/, \mathbf{x}) = I(\mathcal{N}(/0/), \mathbf{x}) = I(/1/, \mathbf{x}) = \mathbf{x}$ by $\mathcal{I}3$.
$(v)\mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}, \mathbf{y}), \mathbf{z}) = \mathcal{I}(\mathcal{N}(\mathcal{I}(\mathcal{N}(\mathbf{x}), \mathbf{y})), \mathbf{z}) = \mathcal{I}(\mathcal{N}(\mathbf{z}), \mathcal{I}(\mathcal{N}(\mathbf{x}), \mathbf{y})) =$
$= \mathcal{I}(\mathcal{N}(\mathbf{x}), \mathcal{I}(\mathcal{N}(\mathbf{z}), \mathbf{y})) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}, \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{z}, \mathbf{y})) = \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{x}, \mathcal{S}_{\mathcal{I},\mathcal{N}}(\mathbf{y}, \mathbf{z}))$ by $\mathcal{I}$5, $\mathcal{I}$9b and (iv).

Therefore, Proposition 5.3.3 is verified. □

Theorem 5.3.1 *For a function $\mathcal{I} : (L_n(U))^2 \rightarrow L_n(U)$ the following are equivalent:*

(i) $\mathcal{I}$ is an n-dimensional $(\mathcal{S}, \mathcal{N})$-implication with a continuous n-DN $\mathcal{N}$.

(ii) $\mathcal{I}$ satisfies $\mathcal{I}1, \mathcal{I}5$ and $\mathcal{N}_{\mathcal{I}}$ is a continuous n-DN $(\mathcal{I}12)$.

And the representation of $(\mathcal{S}, \mathcal{N})$-implication is unified $\mathcal{N}_{\mathcal{I}}(\mathbf{x}) = \mathcal{I}(\mathbf{x}, /0/)$.

($\Leftarrow$) By Proposition 5.3.2, $\mathcal{S}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) = \mathcal{S}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}), \mathbf{y})$ is a t-conorm. Additionally, if $\mathcal{I}$ verifies $\mathcal{I}3, \mathcal{I}5$ and $\mathcal{I}12$, meaning that $\mathcal{N}_{\mathcal{I}}$ is a continuous n-DN, then $\mathcal{I}$ is a n-dimensional fuzzy implication.

($\Rightarrow$) Let $\mathcal{I}$ be an n-dimensional $(\mathcal{S}, \mathcal{N})$-implication based on a t-conorm $\mathcal{S}$ and a continuous n-dimensional negation $\mathcal{N}$. The following is held:

By Proposition 5.3.2(i) .

$\mathcal{I}1$: If $\mathbf{x}_1 \leq \mathbf{x}_2$ then based on monotonicity of $\mathcal{S}$ ($\mathcal{S}4$) and monotonicity of $\mathcal{N}$ ($\mathcal{N}2$),

$$\mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x}, \mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathbf{x}_1), \mathbf{y}) \geq \mathcal{S}(\mathcal{N}(\mathbf{x}_2), \mathbf{y}) = \mathcal{I}_{\mathcal{S},\mathcal{N}}(\mathbf{x}_2, \mathbf{y}).$$

$\mathcal{I}5$: Strainforward from $\mathcal{S}3$.

$\mathcal{I}12$: $\mathcal{I}(\mathbf{x}, /0/) = \mathcal{S}(\mathcal{N}_{\mathcal{I}}(\mathbf{x}), /0/) = \mathcal{N}_{\mathcal{I}}(\mathbf{x})$ by $\mathcal{S}1$, then it is a continuous n-DN.

Proposition 5.3.4 *Let $\mathcal{I}_{S,N}$ be n-DFSI given by Eq.(55). The following holds:*

$$\mathcal{I}_{\widetilde{S_1 \ldots S_n}, \mathcal{N}}(\mathbf{x}, \mathbf{y}) = \widetilde{I_{S_1,N} \ldots I_{S_n,N}}(\mathbf{x}, \mathbf{y}), \tag{56}$$

is an n-dimensional $(\mathcal{S}, \mathcal{N})$ implication.

Proof: For all $\mathbf{x}, \mathbf{y} \in L_n(U)$, we have that

$$\begin{aligned}\mathcal{I}_{\widetilde{S_1 \ldots S_n},N}(\mathbf{x},\mathbf{y}) &= \widetilde{S_1 \ldots S_n}(\widetilde{N}(\mathbf{x}),\mathbf{y})\\ &= (\widetilde{S_1 \ldots S_n}((N(x_n),\ldots,N(x_1)),\mathbf{y}))) = (S_1(N(x_n),y_1),\ldots,S_n(N(x_1),y_n))\\ &= (I_{S_1,N}(x_n,y_1),\ldots,I_{S_n,N}(x_1,y_n)) = \widetilde{I_{S_1,N} \ldots I_{S_n,N}}(\mathbf{x},\mathbf{y}).\end{aligned}$$

Therefore, Proposition 5.3.4 is verified. □

Proposition 5.3.5 *(ZANOTELLI; REISER; BEDREGAL, 2018a) Let I_{S_i,N_i} be $(\mathcal{S},\mathcal{N})$-implication, the $\widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}$ is a n-dimensional fuzzy $(\mathcal{S},\mathcal{N})$-implication.*

Proof: $\Rightarrow$ For $\widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}$ given by Eq.(50) we have that
$\mathcal{I}1$: If $\mathbf{x} \leq \mathbf{z}$ then $(x_1,\ldots,x_n) \leq (z_1,\ldots,z_n)$. Since I_{S_i,N_i} verifies Property $\mathbf{I}1$, when $x_i \leq z_i$ then, by Property $\mathbf{I}1$ and Eq.(50) $I_{S_i,N_i},(x_i,y_i) \geq I_{S_i,N_i}(z_i,y_i)$, for $0 \leq i \leq n$. So,

$$\begin{aligned}\widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(\mathbf{x},\mathbf{y}) &= (I_{S_1,N_1}(x_n,y_1),\ldots,I_{S_n,N_n}(x_1,y_n)) \geq (I_{S_1,N_1}(z_n,y_1),\ldots,I_{S_n,N_n}(z_1,y_n))\\ &= \widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(\mathbf{z},\mathbf{y}).\end{aligned}$$

$\mathcal{I}2$: If $\mathbf{y} \leq \mathbf{z}$ then $(y_1,\ldots,y_n) \leq (z_1,\ldots,z_n)$. Since I_{S_i,N_i} verifies Property $\mathbf{I}2$, when $y_i \leq z_i$ then $I_{S_i,N_i}(x,y_i) \leq I_{S_i,N_i}(x,z_i)$ for $0 \leq i \leq n$. By Property $\mathbf{I}2$ and Eq.(50), we have that

$$\begin{aligned}\widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(\mathbf{x},\mathbf{y}) &= (I_{S_1,N_1}(x_n,y_1),\ldots,I_{S_n,N_n}(x_1,y_n)) \leq (I_{S_1,N_1}(x_n,z_1),\ldots,I_{S_n,N_n}(x_1,z_n))\\ &= \widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(\mathbf{x},\mathbf{z}).\end{aligned}$$

$\mathcal{I}3$: By Property $\mathbf{I}3$ and Eq.(50),we have that

$$\begin{aligned}\widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(/\mathbf{1}/,/\mathbf{y}/) &= (I_{S_1,N_1}(/\mathbf{1}/,y_1),\ldots,I_{S_n,N_n}(/\mathbf{1}/,y_n))\\ &= (S_1(N_1(1),y_1),\ldots,S_n(N_n(1),y_n)) = \mathbf{y}.\end{aligned}$$

$\mathcal{I}5$: By Property $\mathbf{I}5$ and Eq.(50), we have that

$$\begin{aligned}&\widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(\mathbf{x},\widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(\mathbf{y},\mathbf{z}))\\ &= \widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(\mathbf{x},S_1(N_1(y_n),z_1),\ldots,S_n(N_n(y_1),z_n))\\ &= (S_1(N_1(x_n),S_1(N_1(y_n),z_1)),\ldots,S_n(N_n(x_1),S_n(N_n(y_1),z_n)))\\ &= (S_1(N_1(y_n),S_1(N_1(x_n),z_1)),\ldots,S_n(N_n(y_1),S_n(N_n(x_1),z_n)))\\ &= \widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(\mathbf{y},S_1(N_1(x_n),z_1)),\ldots,S_n(N_n(x_1),z_n)))\\ &= \widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(\mathbf{y},\widetilde{I_{S_1,N_1} \ldots I_{S_n,N_n}}(\mathbf{x},\mathbf{z}).\end{aligned}$$

$\mathcal{I}9$: By Property $\mathbf{I}9$ and Eq.(50), we have that

$$\begin{aligned} I_{S_1,N_1}\widetilde{\ldots I}_{S_n,N_n}(\mathcal{N}(\mathbf{y}),\mathcal{N}(\mathbf{x})) &= I_{S_1,N_1}(N_1(y_n),N_1(x_1),\ldots,I_{S_n,N_n}(N_n(y_1),N_n(x_n)) \\ &= (S_1(N_1(N_1(y_n)),N_1(x_1),\ldots,S_n(N_n(N_n(y_1)),N_n(x_n)))) \\ &= I_{S_1,N_1}\widetilde{\ldots I}_{S_n,N_n}(\mathbf{x},\mathbf{y}). \end{aligned}$$

$\Leftarrow$ Analogously, the reverse construction can be proved.
Therefore, Proposition 5.3.5 is verified. □

5.3.2 Conjugation of n-Dimensional $(\mathcal{S},\mathcal{N})$-implication

Next proposition extend the results in (BACZYŃSKI; JAYARAM, 2008).

Proposition 5.3.6 *If $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is an $(\mathcal{S},\mathcal{N})$-implication, then the φ-conjugate of $\mathcal{I}_{\mathcal{S},\mathcal{N}}$ is also an $(\mathcal{S},\mathcal{N})$-implication generated from the φ-conjugate n-DS of $\mathcal{S}$ and the φ-conjugate n-DN of $\mathcal{N}$, which is given as*

$$(\mathcal{I}_{\mathcal{S},\mathcal{N}})_\varphi(\mathbf{x},\mathbf{y}) = \mathcal{I}_{\mathcal{S}_\varphi,\mathcal{N}_\varphi}(\mathbf{x},\mathbf{y}). \tag{57}$$

Proof: Let $\varphi \in Aut(L_n(U))$ and let $\mathcal{S},\mathcal{N}$ be an n-DS and an n-DN, respectively. The functions $\mathcal{S}_\varphi,\mathcal{N}_\varphi$ are a n-DITS and an n-DN, thus

$$\begin{aligned} (\mathcal{I}_{\mathcal{S},\mathcal{N}})_\varphi(\mathbf{x},\mathbf{y}) &= \varphi^{-1}(\mathcal{I}_{\mathcal{S},\mathcal{N}}(\varphi(\mathbf{x}),\varphi(\mathbf{y}))) \\ &= \varphi^{-1}(\widetilde{S_1,\ldots,}S_n(\widetilde{N,\ldots,}N(\varphi(\mathbf{x})),\varphi(\mathbf{y}))) \text{ by Eqs.(33) and (44)} \\ &= \varphi^{-1}(\widetilde{S_1,\ldots,}S_n(\varphi\circ\varphi^{-1}(N(\varphi(x_n)),\ldots,N(\varphi(x_1))),\varphi(\mathbf{y}))) \text{ by Eqs.(30) and (37)} \\ &= \varphi^{-1}(S_1(\varphi(N_\varphi(x_n),\varphi(y_1))),\ldots,S_n(\varphi(N_\varphi(x_1),\varphi(y_n)))) \\ &= S_{\varphi_1}(N_\varphi(x_n),y_1),\ldots,S_{\varphi_n}(N_\varphi(x_1),y_n) = \mathcal{I}_{\mathcal{S}_\varphi,\mathcal{N}_\varphi}(\mathbf{x},\mathbf{y}). \end{aligned}$$

Therefore, Proposition 5.3.6 is verified. □

5.3.3 QL-implications on $L_n(U)$

Definitions below extend results from (BACZYŃSKI; JAYARAM, 2010, Definitions 4.1 and 4.2) to $L_n(U)$.

Definition 5.3.1 *(ZANOTELLI; REISER; BEDREGAL, 2018b, Definition 7) A function $\mathcal{I} : (L_n(U))^2 \to L_n(U)$ is an n-dimensional QL-operator if there exist an n-dimensional t-conorm $\mathcal{S} : (L_n(U))^2 \to L_n(U)$, an n-dimensional negation $\mathcal{N} : L_n(U) \to L_n(U)$ and an n-dimensional t-norm $\mathcal{T} : (L_n(U))^2 \to L_n(U)$ such that*

$$\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(\mathbf{x},\mathbf{y}) = \mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{T}(\mathbf{x},\mathbf{y})). \tag{58}$$

Proposition 5.3.7 *(ZANOTELLI; REISER; BEDREGAL, 2018b, Proposition 9) Let* $\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}\!:\!(L_n(U))^2\!\rightarrow\! L_n(U)$ *be an* n*-dimensional* QL*-operator. Then, the following holds:*

(i) $\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}$ *satisfies* $\mathcal{I}0(a), \mathcal{I}0(b), \mathcal{I}2, \mathcal{I}3$ *and* $\mathcal{I}10$ *properties;*

(ii) $\mathcal{N}(\mathbf{x}) = \mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(\mathbf{x}, /0/)$.

Proof: (i) First, we consider the boundary conditions on $L_n(U)$.

$$\begin{aligned}
\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(/0/,/0/) &= \widetilde{S_1\ldots S_n}((N_1(0),\ldots,N_n(0)),(T_1(0,0),\ldots,T_n(0,0)))\\
&= (S_1(N_1(0),T_1(0,0)),\ldots,S_n(N_n(0),T_n(0,0)))\\
&= (S_1(1,0),\ldots,S_n(1,0)) = (1,\ldots,1) = /1/.\\
\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(/1/,/1/) &= \widetilde{S_1\ldots S_n}(\widetilde{N}(/1/),\widetilde{T_1\ldots T_n}(/1/,/1/))\\
&= (S_1(N_1(1),T_1(1,1)),\ldots,S_n(N_n(1),T_n(1,1)))\\
&= (S_1(0,1),\ldots,S_n(0,1)) = (1,\ldots,1) = /1/.\\
\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(/0/,/1/) &= \widetilde{S_1\ldots S_n}((N_1(0),\ldots,N_n(0)),(T_1(0,1),\ldots,T_n(0,1)))\\
&= (S_1(N_1(0),T_1(0,1)),\ldots,S_n(N_n(0),T_n(0,1)))\\
&= (S_1(1,0),\ldots,S_n(1,0)) = (1,\ldots,1) = /1/.\\
\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(/1/,/0/) &= \widetilde{S_1\ldots S_n}((N_1(1),\ldots,N_n(1)),(T_1(1,0),\ldots,T_n(1,0)))\\
&= (S_1(N_1(1),T_1(1,0)),\ldots,S_n(N_n(1),T_n(1,0)))\\
&= (S_1(0,0),\ldots,S_n(0,0)) = (0,\ldots,0) = /0/.
\end{aligned}$$

Then $\mathcal{I}0$ is verified. Moreover, for $\mathbf{x},\mathbf{y}_1,\mathbf{y}_2 \in L_n(U), \mathbf{y}_1 \le \mathbf{y}_2$, we have that:

$$\begin{aligned}
\mathcal{I}2 : \mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(\mathbf{x},\mathbf{y}) &= \widetilde{S_1\ldots S_n}((N_1(x_n),\ldots N_n(x_1)),(T_1(x_1,y_1),\ldots,T_n(x_n,y_n)))\\
&\le \widetilde{S_1\ldots S_n}((N_1(x_n),\ldots N_n(x_1)),(T_1(x_1,z_1),\ldots,T_n(x_n,z_n)))\\
&= (S_1(N_1(x_n),T_1(x_1,y_1)),\ldots,S_n(N_n(x_1),T_n(x_n,y_n)))\\
&\le (S_1(N_1(x_n),T_1(x_1,z_1)),\ldots,S_n(N_n(x_1),T_n(x_n,z_n))).
\end{aligned}$$

Other two properties are also verified:

$$\begin{aligned}
\mathcal{I}3 : \mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(/1/,\mathbf{y}) &= \widetilde{S_1\ldots S_n}((N_1(1),\ldots,N_n(1)),(T_1(1,y_1),\ldots,T_n(1,y_n)))\\
&= (S_1(N_1(1),T_1(1,y_1)),\ldots,S_n(N_n(1),T_n(1,y_n)))\\
&= (S_1(0,y_1),\ldots,S_n(0,y_n)) = (y_1,\ldots,y_n) = \mathbf{y}.\\
\mathcal{I}10 : \mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(/0/,\mathbf{x}) &= \widetilde{S_1\ldots S_n}((N_1(0),\ldots,N_n(0)),(T_1(0,x_1),\ldots,T_n(0,x_n)))\\
&= (S_1(N_1(0),T_1(0,x_1)),\ldots,S_n(N_n(0),T_n(0,x_n)))\\
&= (S_1(1,0),\ldots,S_n(1,0)) = (1,\ldots,1) = /1/.
\end{aligned}$$

Finally, we have that:

$$(ii)\ \mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(\mathbf{x},/0/) = \mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{T}(\mathbf{x},/0/)) = \mathcal{S}(\mathcal{N}(\mathbf{x}),/0/) = \mathcal{N}(\mathbf{x}).$$

and therefore, Proposition 5.3.7 holds. □

Remark 5.3.2 *Based on results from (BACZYŃSKI; JAYARAM, 2010, Theorem 2), by taking $\mathcal{N}$ as strong n-DN and applying the distributivity of $\mathcal{S}_M$ with respect to $\mathcal{T}_M$, and vice-versa, we have that for all $\mathbf{x},\mathbf{y} \in L_n(U)$ such that $\mathbf{x} \leq \mathbf{y}$, the following holds:*

$$\begin{aligned}
\mathcal{I}_{\widetilde{S_M},\mathcal{N},\widetilde{T_M}}(\mathbf{x},\mathcal{I}_{\widetilde{S_M},\mathcal{N},\widetilde{T_M}}(\mathbf{x},\mathbf{y})) &= \mathcal{I}_{\widetilde{S_M},\mathcal{N},\widetilde{T_M}}(\widetilde{S_M}(\mathcal{N}(\mathbf{x}),\widetilde{T_M}(\mathbf{x},\mathbf{y}))\\
&= (\widetilde{S_M}(\mathcal{N}(\mathbf{x}),\widetilde{T_M}(\mathbf{x},\widetilde{S_M}(\mathcal{N}(\mathbf{x}),\widetilde{T_M}(\mathbf{x},\mathbf{y}))\\
&= (\widetilde{S_M}(\mathcal{N}(\mathbf{x}),\widetilde{T_M}(\mathbf{x},\widetilde{S_M}(\mathcal{N}(\mathbf{x}),\mathbf{y})))\\
&= \widetilde{S_M}(\mathcal{N}(\mathbf{x}),\widetilde{T_M}(\mathbf{x},\mathbf{y}) = \mathcal{I}_{\widetilde{S_M},\mathcal{N},\widetilde{T_M}}(\mathbf{x},\mathbf{y}).
\end{aligned}$$

Meaning that the QL-implicator $\mathcal{I}_{\widetilde{S_M},\mathcal{N},\widetilde{T_M}}$ is an example of QL-implication.

Proposition 5.3.8 *(ZANOTELLI; REISER; BEDREGAL, 2018b) Let $(\mathcal{S},\mathcal{N})$ be a pair satisfying property **LEM** on $L_n(U)$. Then $\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}$ verifies $\mathcal{I}4$.*

Proof: For all $\mathbf{x} \in L_n(U)$, the following holds:

$$\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(/\mathbf{x}/,/\mathbf{1}/) = \mathcal{S}(\mathcal{N}(\mathbf{x}),\mathcal{T}(\mathbf{x},/\mathbf{1}/)) = \mathcal{S}(\mathcal{N}(\mathbf{x}),/\mathbf{1}/) = /\mathbf{1}/.$$

□

5.3.3.1 QL-implicator on $L_n(U)$ obtained by QL-implicators on U

In this section, we consider the conditions under which a QL-implicator on $L_n(U)$ can be obtained by a family of QL-implicators on U, meaning that it can be representable by such family.

Proposition 5.3.9 *(ZANOTELLI; REISER; BEDREGAL, 2018b) An n-dimensional QL-implicator $\mathcal{I}_{\widetilde{S_1\ldots S_n},\widetilde{N},\widetilde{T_1\ldots T_n}} : (L_n(U))^2 \to L_n(U)$ can be expressed as follows:*

$$\mathcal{I}_{\widetilde{S_1\ldots S_n},\widetilde{N},\widetilde{T_1\ldots T_n}}(/x/,\mathbf{y}) = \widetilde{I_{S_1,N,T_1}\ldots I_{S_n,N,T_n}}(/x/,\mathbf{y}). \tag{59}$$

Proof: For all $/x/, \mathbf{y} \in L_n(U)$, we have that:

$$\begin{aligned}
\mathcal{I}_{\widetilde{S_1 \ldots S_n}, \widetilde{N}, \widetilde{T_1 \ldots T_n}}(/x/, \mathbf{y}) &= \widetilde{S_1 \ldots S_n}(\widetilde{N}(/x/), \widetilde{T_1 \ldots T_n}(/x/, \mathbf{y})\\
&= \widetilde{S_1 \ldots S_n}((N(x) \ldots N(x)), T_1(x, y_1) \ldots T_n(x, y_n)))\\
&= (S_1(N(x), T_1(x, y_1)), \ldots, S_n(N(x), T_n(x, y_n)))\\
&= (I_{S_1,N,T_1}(x, y_1), \ldots, I_{S_n,N,T_n}(x, y_n))\\
&= \widetilde{I_{S_1,N,T_1} \ldots I_{S_n,N,T_n}}(/x/, \mathbf{y}).
\end{aligned}$$

Concluding, Proposition 5.3.9 is verified. □

Example 5.3.1 *Considering the following QL-implications on U: $I_{LK} = I_{S_{LK},N_S,T_M}$, $I_{RC} = I_{S_{LK},N_S,T_P}$ and $I_{KD} = I_{S_{LK},N_S,T_{LK}}$, the QL-implication $\widetilde{I_{LK}, I_{RC}, I_{KD}} : (L_n(U))^2 \to L_n(U)$ can be expressed as:*

$$\mathcal{I}_3(/x/, \mathbf{y}) = \widetilde{I_{LK}, I_{RC}, I_{KD}}(/x/, \mathbf{y}) = \mathcal{I}_{\tilde{S}_{LK}, \tilde{N_S}, \widetilde{T_M, T_P, T_{LK}}}(/x/, \mathbf{y}).$$

5.3.3.2 Conjugate QL-implicators on $L_n(U)$

Based on results from Eq.(28), the action of an φ-automorphism on a QL-implicator is considered in the following.

Proposition 5.3.10 *(ZANOTELLI; REISER; BEDREGAL, 2018b) Consider $\varphi \in Aut(L_n(U))$. If $\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}$ is a QL-implicator, the corresponding φ-conjugate QL-implicator is given as*

$$\mathcal{I}^{\varphi}_{\mathcal{S},\mathcal{N},\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \mathcal{S}^{\varphi}(\mathcal{N}^{\varphi}(\mathbf{x}), \mathcal{T}^{\varphi}(\mathbf{x}, \mathbf{y})). \tag{60}$$

Proof: Let $\varphi \in Aut(L_n(U))$ and the functions $\mathcal{S}, \mathcal{N}$ and $\mathcal{T}$ be an n-dimensional t-conorm, an n-dimensional fuzzy negation, and an n-dimensional t-norm, respectively. For all $\mathbf{x}, \mathbf{y} \in L_n(U)$, the following holds:

$$\begin{aligned}
\mathcal{I}^{\varphi}_{\mathcal{S},\mathcal{N},\mathcal{T}}(\mathbf{x}, \mathbf{y}) &= \varphi^{-1}(\mathcal{I}_{\mathcal{S},\mathcal{N},\mathcal{T}}(\varphi(\mathbf{x}), \varphi(\mathbf{y})))) \text{ (by Eq.(28))}\\
&= \varphi^{-1}(\mathcal{S}(\mathcal{N}(\varphi(\mathbf{x})), \mathcal{T}(\varphi(\mathbf{x}), \varphi(\mathbf{y})))) \text{ (by Eq.(58))}\\
&= \varphi^{-1}(\mathcal{S}(\varphi \circ \varphi^{-1}(\mathcal{N}(\varphi(\mathbf{x}))), \varphi \circ \varphi^{-1}(\mathcal{T}(\varphi(\mathbf{x}), \varphi(\mathbf{y})))))\\
&= \varphi^{-1}(\mathcal{S}(\varphi(\mathcal{N}^{\varphi}(\mathbf{x})), \varphi(\mathcal{T}^{\varphi}(\mathbf{x}, \mathbf{y})))) = \mathcal{S}^{\varphi}(\mathcal{N}^{\varphi}(\mathbf{x}), \mathcal{T}^{\varphi}(\mathbf{x}, \mathbf{y})).
\end{aligned}$$

Therefore, Proposition 5.3.10 holds. □

Corollary 5.3.1 *(ZANOTELLI; REISER; BEDREGAL, 2018b) Let $T_1, \ldots, T_n(S_1, \ldots, S_n) : U^2 \to U$ be t-norm (t-conorm) such that $T_1 \leq \ldots \leq T_n(S_1 \leq$*

$\ldots \leq S_n)$*, according with Theorem 5.1.3. By considering* φ *as a* ϕ*-representable automorphism on* $Aut(L_n(U))$*, the expression Eq.(60) of the* $\tilde{\phi}$*-conjugate of a* QL*-implicator* $I_{\mathcal{S},\tilde{N},\mathcal{T}}$ *can be reduced to*

$$\mathcal{I}^{\tilde{\phi}}_{\widetilde{S_1\ldots,S_n},\widetilde{N},\widetilde{T_1,\ldots,T_n}}(\mathbf{x},\mathbf{y}) = \mathcal{I}_{\widetilde{S_1^{\phi}\ldots,S_n^{\phi}},\widetilde{N^{\phi}},\widetilde{T_1^{\phi},\ldots,T_n^{\phi}}}(\mathbf{x},\mathbf{y}).$$

Proof: Straightforward from results of Proposition 5.3.10. □

Example 5.3.2 *Let* $\mathcal{I}_3 \equiv (I_{KD}, I_{RC}, I_{LK})$ *be a* QL*-implications on* $L_n(u)$ *given in Example 2. Therefore, by considering* $\varphi = \tilde{\phi}$*, we have that* $\mathcal{I}_3^{\tilde{\phi}} \equiv (I^{\tilde{\phi}}_{KD}, I^{\tilde{\phi}}_{RC}, I^{\tilde{\phi}}_{LK})$ *is also a* QL*-implicator on* $L_n(u)$*. Moreover, when* $\phi = x^n$*, we have that*

$$\mathcal{I}_3^{\tilde{\phi}}(/x/,\mathbf{y}) = (\max(1-x^n, y_1^n), 1-x^n+x^ny_2^n, \min(1, 1-x^n+y_3^n)).$$

6 RESIDUAL FUZZY IMPLICATIONS ON $L_n(U)$

According to (BEDREGAL et al., 2007), the class of R-implications plays an important role in FL. In a broad sense, it is frequently applied to fuzzy control, analysis of vagueness in natural language and techniques of soft-computing. In the narrow sense, the extension of theoretical research of R-implications contributes to a branch of many valued logic enabling the investigation of deep logical questions involving the residuation property (ALCALDE; BURUSCO; FUENTES-GONZÁLEZ, 2005; BACZYŃSKI, 2004; RUIZ; TORRENS, 2004).

The residuation principle (RP) is also important in operational semantics of programming languages (ESPANA; ESTRUCH, 2004). In addition, the R-implication derived from a left continuous t-norm is a general logical framework in mathematic morphology (BLOCH, 2009), defining the fuzzy morphological operators: fuzzy dilation and fuzzy erosion. Another important application of residual implications in image processing (SHI; GASSE; KERRE, 2013; BARRENECHEA et al., 2011) is concerned with subsethood and similarity measures (SANTOS et al., 2019), performing the comparison of digital images represented by multi-valued fuzzy sets. In addition, it is frequently applied to achieve solutions for MCDM.

Moreover, the essential idea of residuation, in connection with formal fuzzy logic, plays a fundamental role in automata theory (FARIAS et al., 2016). To sum up, the concept of residuation concerning the theory of residuated lattices constitutes the principal structure of several classes of algebraic logics (SANTOS; BEDREGAL, 2008).

6.1 R-implications on $L_n(U)$

The main properties of R-implications extended from U to $L_n(U)$ are discussed below.

Definition 6.1.1 *A function $\mathcal{I}_{\mathcal{T}} : (L_n(U))^2 \to L_n(U)$ is called an n-dimensional R-implication (n-DRI) if there exists n-DT $\mathcal{T} : (L_n(U))^2 \to L_n(U)$ such that*

$$\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \le \mathbf{y}\}. \tag{61}$$

Next proposition extends results from (BACZYŃSKI; JAYARAM, 2008a, Theorem 5.5).

Proposition 6.1.1 *If $\mathcal{T}$ is an n-DT then $\mathcal{I}_{\mathcal{T}} \in \mathcal{I}(L_n(U))$. Moreover, it verifies $\mathcal{I}0$, $\mathcal{I}1, \mathcal{I}2, \mathcal{I}3$ and $\mathcal{I}14$. In addition, it also verifies $\mathcal{I}12$, meaning that its natural negation $\mathcal{N}_{\mathcal{I}}$ coincides with the $\mathcal{N}_{\mathcal{T}}$ given in Eq.(42).*

Proof: Let $\mathcal{T}$ be an n-DT and $\mathcal{I}_{\mathcal{T}}$ be the function defined by Eq.(61). Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$.
$\mathcal{I}0$: The boundary conditions hold as follows:

$$\mathcal{I}_{\mathcal{T}}(/1/, /1/) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(/1/, \mathbf{z}) = \mathbf{z} \leq /1/\} = /1/;$$
$$\mathcal{I}_{\mathcal{T}}(/0/, /1/) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(/0/, \mathbf{z}) = /0/ \leq /1/\} = /1/;$$
$$\mathcal{I}_{\mathcal{T}}(/0/, /0/) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(/0/, \mathbf{z}) = /0/ \leq /0/\} = /1/;$$
$$\mathcal{I}_{\mathcal{T}}(/1/, /0/) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(/1/, \mathbf{z}) = \mathbf{z} \leq /0/\} = /0/.$$

$\mathcal{I}1$: Let $\mathbf{x}_1, \mathbf{x}_2 \in L_n(U)$. Based on monotonicity of $\mathcal{T}$, when $\mathbf{x}_1 \leq \mathbf{x}_2$, taking $\mathbf{z} \in L_n(U)$ such that $\mathcal{T}(\mathbf{x}_2, \mathbf{z}) \leq \mathbf{y}$ we should have that $\mathcal{T}(\mathbf{x}_1, \mathbf{z}) \leq \mathbf{y}$. Thus, the inclusion

$$\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}_1, \mathbf{z}) \leq \mathbf{y}\} \supset \{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}_2, \mathbf{z}) \leq \mathbf{y}\},$$

is verified. Then, it implies that

$\mathcal{I}_{\mathcal{T}}(\mathbf{x}_1, \mathbf{y}) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}_1, \mathbf{z}) \leq \mathbf{y}\}$ and so, $\mathcal{I}_{\mathcal{T}}(\mathbf{x}_1, \mathbf{y}) \geq \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}_2, \mathbf{z}) \leq \mathbf{y}\}$.

Therefore, $\mathcal{I}_{\mathcal{T}}(\mathbf{x}_1, \mathbf{y}) \geq \mathcal{I}_{\mathcal{T}}(\mathbf{x}_2, \mathbf{y})$.
$\mathcal{I}2$: Analogous to $\mathcal{I}1$.
$\mathcal{I}3$: $\mathcal{I}_{\mathcal{T}}(/1/, \mathbf{y}) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(/1/, \mathbf{z}) = \mathbf{z} \leq \mathbf{y}\} = \mathbf{y}$.
$\mathcal{I}12$: $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, /0/) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq /0/\} = \mathcal{N}_{\mathcal{T}}(\mathbf{x})$, by Eq.(42), which is an n-DN according to (BEDREGAL et al., 2012).
$\mathcal{I}14$: $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{x}) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{x}\} = /1/$.
Concluding, Proposition 6.1.1 is verified. □

Based on results presented in Proposition 6.1.1, the natural negation $\mathcal{N}_{\mathcal{I}}$ coincides with the $\mathcal{N}_{\mathcal{T}}$ given in Eq.(42) when $\mathcal{I}$ is an n-DRI and $\mathcal{T}$ their underlying n-DT on $L_n(U)$.

Corollary 6.1.1 *There is no representable n-DI which is an n-DRI.*

Proof: Straightforward from Propositions. 5.2.8 and 6.1.1. □

6.2 Residuation property on $L_n(U)$

The following results show the necessary and sufficient conditions under which the pair of functions $(\mathcal{I}_{\mathcal{T}}, \mathcal{T})$ verifies the residuation property on $\mathcal{L}_n(U)$. In the following theorem, results from (BACZYŃSKI; JAYARAM, 2008b) are extended for n-dimensional approach.

Theorem 6.2.1 *Let $\mathcal{T}$ be an n-DT. The following statements are equivalent:*
1. *$\mathcal{T}$ safisties the left-continuity property:*

$$\mathcal{LC}: \lim_{n\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_n) = \mathcal{T}(\mathbf{x}, \lim_{n\to\infty} \mathbf{y}_n); \tag{62}$$

2. *$(\mathcal{T}, \mathcal{I}_{\mathcal{T}})$ is an adjoint pair, verifying the residuation property:*

$$\mathcal{RP}: \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y} \Leftrightarrow \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) \geq \mathbf{z}; \tag{63}$$

3. *$\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \max\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$.*

Proof: $(\mathbf{1}.) \Rightarrow (\mathbf{2}.)$ Let $\mathcal{T}$ be a left-continuous n-DT and assume that $\mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathbf{y}$, for some $\mathbf{x}, \mathbf{y}, \mathbf{t} \in L_n(U)$. So, we have that $\mathbf{t} \in \{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$. Therefore, $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) \geq \mathbf{t}$. Conversely, assuming that $\mathbf{t} \leq \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y})$, for some $\mathbf{x}, \mathbf{y}, \mathbf{t} \in L_n(U)$. Both situations need to be considered:
(1) If $\mathbf{t} < \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y})$, then there exists some $\mathbf{z}' \in L_n(U)$, $\mathbf{t} < \mathbf{z}'$ such that $\mathcal{T}(\mathbf{x}, \mathbf{z}') \leq \mathbf{y}$. By property $\mathcal{T}3$, $\mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathcal{T}(\mathbf{x}, \mathbf{z}') \leq \mathbf{y}$, implying that $\mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathbf{y}$.
(2) If $\mathbf{t} = \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y})$. In this case, we observe that

- **(i)** either $\mathbf{t} \in \{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$ implying that $\mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathbf{y}$;

- **(ii)** or $\mathbf{t} \notin \{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$. Since

  $$\mathbf{t} = \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\},$$

  there exists an increasing sequence $(\mathbf{z}_i)_{i\in\mathbb{N}}$ such that $\mathbf{z}_i < \mathbf{t}, \mathcal{T}(\mathbf{x}, \mathbf{z}_i) \leq \mathbf{y}$ and $\lim_{i\to\infty} \mathbf{z}_i = \mathbf{t}, \forall i \in \mathbb{N}$. Since $\mathcal{T}$ is a left-continuous n-DT, it holds that

  $$\mathcal{T}(\mathbf{x}, \mathbf{t}) = \mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{z}_i) = \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{z}_i) \leq \mathbf{y}.$$

Therefore, from all above cases, (1), (2i) and (2ii) we have that if $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) \geq \mathbf{t}$ then $\mathcal{T}(\mathbf{x}, \mathbf{z}_i) \leq \mathbf{y}$.

$(\mathbf{2}.) \Rightarrow (\mathbf{3}.)$ Assuming that $\mathcal{T}$ and $\mathcal{I}_{\mathcal{T}}$ verifies $\mathcal{RP}$, since $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) \leq \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y})$, we obtain that $\mathcal{T}(\mathbf{x}, \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y})) \leq \mathbf{y}$ and then, $\mathcal{I}_{\mathcal{T}} \in \{\mathbf{z} : \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$. Therefore, we conclude that $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \max\{\mathbf{z} : \mathcal{T}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\}$.

$(\mathbf{3}.) \Rightarrow (\mathbf{1}.)$ Let $\mathcal{T}$ be a n-DT and $(\mathbf{y}_i)_{i\in\mathbb{N}}$ be an increasing n-DS converging to $\mathbf{y}$. Then, from the monotonicity of $\mathcal{T}$, we have the inequality,

$$\mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i) \geq \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i).$$

Let $\mathbf{y} = \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i)$, implying that $\mathcal{T}(\mathbf{x}, \mathbf{y}_i) \leq \mathcal{T}(\mathbf{x}, \mathbf{y}) \leq \mathbf{y}$, $\forall i \in \mathbb{N}$. Then, $\mathbf{y}_i \in \{\mathbf{t} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathbf{y}\}, \forall i \in \mathbb{N}$. Based on such results we obtain that $\mathbf{y}_i \leq \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y})$, $\forall i \in \mathbb{N}$. So, we can deduce that

$$\lim_{i\to\infty} \mathbf{y}_i \leq \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}).$$

Again, by the monotonicity of $\mathcal{T}$ on $L_n(U)$ we obtain that

$$\mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i) \leq \mathcal{T}(\mathbf{x}, \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y})) \leq \mathbf{y} = \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i),$$

meaning that $\lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i) \geq \mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i)$. Therefore, $\mathcal{T}(\mathbf{x}, \lim_{i\to\infty} \mathbf{y}_i) = \lim_{i\to\infty} \mathcal{T}(\mathbf{x}, \mathbf{y}_i)$.
Concluding, Theorem 6.2.1 is verified. □

Proposition 6.2.1 *If $\mathcal{T}$ is a left-continuous n-DT then $\mathcal{I}_{\mathcal{T}}$ is an n-DI satisfying $\mathcal{I}5$ and $\mathcal{I}13$. Moreover, $\mathcal{I}_{\mathcal{T}}$ is left-continuous w.r.t the first variable and right-continuous w.r.t. the second variable.*

Proof: From Proposition 6.1.1 we know that $\mathcal{I}_{\mathcal{T}}$ is an n-DI.

$\mathcal{I}5$: Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$, by Theorem 6.2.1, item (ii), the pair ($\mathcal{T}, \mathcal{I}_{\mathcal{T}}$) satisfies $\mathcal{RP}$. So, the next holds:

$$\begin{aligned}
\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathcal{I}_{\mathcal{T}}(\mathbf{y}, \mathbf{z})) &= \max\{\mathbf{t} \in L_n(U) : \mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathcal{I}_{\mathcal{T}}(\mathbf{y}, \mathbf{z})\}; \text{(by Eq.(61))}\\
&= \max\{\mathbf{t} \in L_n(U) : \mathcal{T}(\mathbf{y}, \mathcal{T}(\mathbf{x}, \mathbf{t})) \leq \mathbf{z}\};\ \text{(by } \mathcal{RP})\\
&= \max\{\mathbf{t} \in L_n(U) : \mathcal{T}(\mathbf{x}, \mathcal{T}(\mathbf{y}, \mathbf{t}) \leq \mathbf{z}\};\ \text{(by } \mathcal{T}2,\ \mathcal{T}4)\\
&= \max\{\mathbf{t} \in L_n(U) : \mathcal{T}(\mathbf{y}, \mathbf{t}) \leq \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{z})\};\\
&= \ \mathcal{I}_{\mathcal{T}}(\mathbf{y}, \mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{z}))\text{(by } \mathcal{RP} \text{ and Eq.(61))}.
\end{aligned}$$

$\mathcal{I}13$: Let $\mathbf{x}, \mathbf{y} \in L_n(U)$. If $\mathbf{x} \leq \mathbf{y}$, then $\mathcal{T}(\mathbf{x}, /1/) = \mathbf{x} \leq \mathbf{y}$, by $\mathcal{RP}$ $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = /1/$. Conversely, if $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = /1/$, then because of $\mathcal{RP}$ we obtain that $\mathcal{T}(\mathbf{x}, /1/) \leq \mathbf{y}$ meaning that $\mathbf{x} \leq \mathbf{y}$.

($\mathcal{LC}$): Now, suppose that $\mathcal{I}_{\mathcal{T}}$ is not left-continuous w.r.t. the first variable in some point $(\mathbf{x}_0, \mathbf{y}_0) \in L_n(U) \times L_n(U)$. Since $\mathcal{I}_{\mathcal{T}}$ satisfies $\mathcal{I}1$, there exist $\mathbf{a}, \mathbf{b} \in L_n(U)$, $\mathbf{a} > \mathbf{b}$ such that $\mathcal{I}_{\mathcal{T}}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{b}$ and $\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}_0) \geq \mathbf{a}$, for all $\mathbf{x} > \mathbf{x}_0$. By Property $\mathcal{RP}$, we obtain that $\mathcal{T}(\mathbf{x}, \mathbf{a}) \leq \mathbf{y}_0$, for all $\mathbf{x} < \mathbf{x}_0$. Since $\mathcal{T}$ is left-continuous, in the limit $\mathbf{x} \to \mathbf{x}_0$ we have $\mathcal{T}(\mathbf{x}_0, \mathbf{a}) \leq \mathbf{y}_0$. From $\mathcal{RP}$, $\mathbf{b} = \mathcal{I}_{\mathcal{T}}(\mathbf{x}_0, \mathbf{y}_0) \geq \mathbf{a}$, which is a contradiction w.r.t. the hypothesis that $\mathbf{a} > \mathbf{b}$. So, $\mathcal{I}_{\mathcal{T}}$ is a left-continuous function w.r.t. the first variable.

($\mathcal{RC}$): Finally, assuming that $\mathcal{I}_{\mathcal{T}}$ is not right-continuous w.r.t. the second variable in some point $(\mathbf{x}_0, \mathbf{y}_0) \in L_n(U) \times L_n(U)$. Since $\mathcal{I}_{\mathcal{T}}$ satisfies $\mathcal{I}2$, there exist $\mathbf{a}, \mathbf{b} \in L_n(U)$ such that $\mathbf{a} > \mathbf{b}$ and $\mathcal{I}_{\mathcal{T}}(\mathbf{x}_0, \mathbf{y}) \geq \mathbf{a}$, for all $\mathbf{y} > \mathbf{y}_0$ and $\mathcal{I}_{\mathcal{T}}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{b}$. By $\mathcal{RP}$, we get

$\mathcal{T}(\mathbf{x}_0, \mathbf{a}) \leq \mathbf{y}$, for all $\mathbf{y} > \mathbf{y}_0$. Then, in the limit $\mathbf{y} \to \mathbf{y}_0$ we have $\mathcal{T}(\mathbf{x}_0, \mathbf{a}) \leq \mathbf{y}_0$. Analogously, from $\mathcal{RP}$, we obtain $b = \mathcal{I}_{\mathcal{T}}(\mathbf{x}_0, \mathbf{y}_0) \geq \mathbf{a}$, contradicting that $\mathbf{a} > \mathbf{b}$. Concluding, $\mathcal{I}_{\mathcal{T}}$ is a right-continuous function w.r.t. the second variable.

Therefore, Proposition 6.2.1 is verified. □

6.3 Conjugation of R-implications on $L_n(U)$

In the following theorem, results from (BACZYŃSKI; JAYARAM, 2008a, Proposition 2.5.10) are extended, showing how n-dimensional automorphisms act on an n-DRI, generating a new n-DRI.

Theorem 6.3.1 *Let $\phi : L_n(U) \to L_n(U)$ be an n-DA and $\mathcal{I}_{\mathcal{T}} : (L_n(U))^2 \to L_n(U)$ be an n-DRI. Then the mapping $\mathcal{I}^{\phi}_{\mathcal{T}} : (L_n(U))^2 \to L_n(U)$ is also an n-DRI defined by*

$$\mathcal{I}^{\phi}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \mathcal{I}_{\mathcal{T}^{\phi}}(\mathbf{x}, \mathbf{y}). \tag{64}$$

Proof: From Proposition 4.5.4, $\mathcal{T}^{\phi} : (L_n(U))^2 \to L_n(U)$ is an n-DT implying that $\mathcal{I}_{\mathcal{T}^{\phi}} : (L_n(U))^2 \to L_n(U)$ is also an n-DRI. By the continuity of bijection ϕ, for $\mathbf{x}, \mathbf{y} \in L_n(U)$, by Eqs.(28),(43) and (61) it holds that:

$$\begin{aligned}
\mathcal{I}^{\phi}_{\mathcal{T}}(\mathbf{x}, \mathbf{y}) = \phi^{-1}(\mathcal{I}_{\mathcal{T}}(\phi(\mathbf{x}), \phi(\mathbf{y}))) &= \phi^{-1}(sup\{\phi(\mathbf{z}) \in L_n(U) \colon \mathcal{T}(\phi(\mathbf{x}), \phi(\mathbf{z})) \leq \phi(\mathbf{y})\}) \\
&= sup\{\phi^{-1}(\phi(\mathbf{z})) \in L_n(U) \colon \phi^{-1}(\mathcal{T}(\phi(\mathbf{x})), \phi(\mathbf{z})) \leq \phi^{-1}(\phi(\mathbf{y}))\} \\
&= sup\{\mathbf{z} \in L_n(U) \colon \mathcal{T}^{\phi}(\mathbf{x}, \mathbf{z}) \leq \mathbf{y}\} = \mathcal{I}_{\mathcal{T}^{\phi}}(\mathbf{x}, \mathbf{y}).
\end{aligned}$$

Therefore, Theorem 6.3.1 is verified. □

Corollary 6.3.1 *Let $\phi : L_n(U) \to L_n(U)$ be an n-DA and $\mathcal{I}_{\mathcal{T}} : (L_n(U))^2 \to L_n(U)$ be an n-DRI. Properties $\mathcal{I}3$, $\mathcal{I}12$ and $\mathcal{I}14$ are invariant under the conjugate ϕ-operator $\mathcal{I}^{\phi}_{\mathcal{T}} : (L_n(U))^2 \to L_n(U)$. In addition, if $\mathcal{T}$ is left-continuous then an n-DI $\mathcal{I}^{\phi}_{\mathcal{T}}$ satisfies the following properties: $\mathcal{I}5$, $\mathcal{I}11$ and $\mathcal{I}13(a)$.*

Proof: It follows from Propositions 5.1.4, 6.1.1 and 6.2.1. □

6.4 Characterizing R-implications on $L_n(U)$

We present the characterization of n-DRI, obtained from left-continuous n-DT considering the method of obtaining n-DT from n-dimensional fuzzy implications by a residuation principle. Since each $\mathcal{I} \in \mathcal{I}(L_n(U))$ verifies the right boundary condition, meaning that $\mathcal{I}(\mathbf{x}, /1/) = /1/$, the function $\mathcal{T}_{\mathcal{I}} : (L_n(U))^2 \to L_n(U)$,

$$\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) = inf\{\mathbf{t} \in L_n(U) \colon \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}\}, \tag{65}$$

is a well-defined function on $L_n(U)$.

Remark 6.4.1 *Let $\widetilde{I_{RC}}$ be the n-dimensional extension of the Reichenbach fuzzy implication given in Example 3.3.1. For $\mathbf{x} > 0$ we obtain $\mathcal{T}_{\widetilde{I_{RC}}}(\mathbf{x}, /1/) = /1/$, meaning that $\mathcal{T}_{\widetilde{I_{RC}}}$ does not satisfy $\mathcal{T}14$. Concluding, Eq.(65) does not always generate an n-DT.*

Remark 6.4.2 *According to Corollary 6.1.1 the function $\widetilde{I_{LK}}$, reported in Example 3.3.1, does not verify the conditions defining an n-DRI. Additionally, let T_{LK} be the Łukasiewicz t-norm. One can easily verify that $\mathcal{I}_{\widetilde{T_{LK}}}$ does not coincide to $\mathcal{I}_{LK}$, the n-dimensional extension of I_{LK} .*

Proposition 6.4.1 *For $\mathcal{I} \in \mathcal{I}(L_n(U))$ the following statements are equivalent:*
$\mathbf{1}$. *$\mathcal{I}$ is right-continuous w.r.t. the second variable.*
$\mathbf{2}$. *$(\mathcal{T}_{\mathcal{I}}, \mathcal{I})$ is an adjoint pair.*
$\mathbf{3}$. *The infimum in Eq.(65) is the minimum, i.e.,*

$$\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) = \min\{\mathbf{t} \in L_n(U) \colon \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}\}, \tag{66}$$

where the right side exists for all $\mathbf{x}, \mathbf{y} \in L_n(U)$.

Proof: $(\mathbf{1}.) \Rightarrow (\mathbf{2}.)$ Suppose firstly that $\mathcal{I}$ is a fuzzy implication which is right-continuous w.r.t. the second variable and also that $\mathcal{I}(\mathbf{x}, \mathbf{z}) \geq \mathbf{y}$ for some $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$. This implies that $\mathbf{z} \in \{\mathbf{t} \in L_n(U) | \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}\}$ and hence $\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) \leq \mathbf{z}$. Conversely, whenever $\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) \leq \mathbf{z}$, for each $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$, two cases are considered:

(i) If $\mathbf{z} > \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y})$, then because $\mathcal{I}$ is right continuous there exists some $\mathbf{t}' < \mathbf{z}$ such that $\mathcal{I}(\mathbf{x}, \mathbf{t}') \geq \mathbf{y}$. So, from $\mathcal{I}2$ we have $\mathcal{I}(\mathbf{x}, \mathbf{z}) \geq \mathcal{I}(\mathbf{x}, \mathbf{t}')$. Therefore, $\mathcal{I}(\mathbf{x}, \mathbf{z}) \geq \mathbf{y}$.

(ii) If $\mathbf{z} = \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) = \inf\{\mathbf{t} \in L_n(U) \colon \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}\}$, then either $\mathbf{z} \in \{\mathbf{t} \in L_n(U) \colon \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}\}$ and so, $\mathcal{I}(\mathbf{x}, \mathbf{z}) \geq \mathbf{y}$; or $\mathbf{z} \notin \{\mathbf{t} \in L_n(U) \colon \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}\}$ but $\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) = \mathbf{z}$ and so, there exists a decreasing sequence $(\mathbf{t}_i)_{i \in \mathbb{N}}$ such that $\mathbf{t}_i > \mathbf{z}, \mathcal{I}(\mathbf{x}, \mathbf{t}_i) \geq \mathbf{y} \forall i \in \mathbb{N}$ and $\lim_{i \to \infty} \mathbf{t}_i = \mathbf{t}$. Therefore, by the right-continuity of $\mathcal{I}$ w.r.t. the second variable, the following is verified:

$$\mathcal{I}(\mathbf{x}, \mathbf{z}) = \mathcal{I}(\mathbf{x}, \lim_{i \to \infty} \mathbf{t}_i) = \lim_{i \to \infty} \mathcal{I}(\mathbf{x}, \mathbf{t}_i) \geq \mathbf{y}.$$

$(\mathbf{2}.) \Rightarrow (\mathbf{3}.)$ Assuming that $(\mathcal{T}_{\mathcal{I}}, \mathcal{I})$ is an adjoint pair. Since $\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) \leq \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y})$, one has that $\mathcal{I}(\mathbf{x}, \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y})) \geq \mathbf{y}$. By the definition of $\mathcal{T}_{\mathcal{I}}$ the infimum in Eq.(65) is the minimum.
$(\mathbf{3}.) \Rightarrow (\mathbf{1}.)$ From the monotonicity of $\mathcal{I}$ w.r.t the second variable, the following inequality holds:

$$\mathcal{I}(\mathbf{x}, \inf_{\mathbf{s} \in \mathcal{S}} \mathbf{y}_{\mathbf{s}}) \leq \inf_{\mathbf{s} \in \mathcal{S}} \mathcal{I}(\mathbf{x}, \mathbf{y}_{\mathbf{s}}). \tag{67}$$

When $\mathbf{y} = \inf_{\mathbf{s}\in\mathcal{S}} \mathcal{I}(\mathbf{x}, \mathbf{y_s})$ then $\mathcal{I}(\mathbf{x}, \mathbf{y_s}) \geq \mathbf{y}, \forall \mathbf{s} \in \mathcal{S}$. Thus, $\mathbf{y_s} \in \{\mathbf{t} \in L_n(U) | \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}\}$, $\forall \mathbf{s} \in \mathcal{S}$. So, $\mathbf{y_s} \geq \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y})$, $\forall \mathbf{s} \in \mathcal{S}$ meaning that $\inf_{\mathbf{s}\in\mathcal{S}} \mathbf{y_s} \geq \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y})$. In addition, by Eq.(65) and $\mathcal{I}2$, it holds that

$$\mathcal{I}(\mathbf{x}, \inf_{\mathbf{s}\in\mathcal{S}} \mathbf{y_s}) \geq \mathcal{I}(\mathbf{x}, \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y})) \geq \mathbf{y} = \inf_{\mathbf{s}\in\mathcal{S}} \mathcal{I}(\mathbf{x}, \mathbf{y_s}). \tag{68}$$

From the above inequalities in Eqs.(67) and (68), we obtain the following result:

$$\mathcal{I}(\mathbf{x}, \inf_{\mathbf{s}\in\mathcal{S}} \mathbf{y_s}) = \inf_{\mathbf{s}\in\mathcal{S}} \mathcal{I}(\mathbf{x}, \mathbf{y_s}), \forall \mathbf{x}, \mathbf{y_s} \in L_n(U), \mathbf{s} \in \mathcal{S}.$$

Meaning that $\mathcal{I}$ is right-continuous w.r.t. the second variable. Therefore, Proposition 6.4.1 is verified. □

Theorem 6.4.1 *If a function $\mathcal{I} : (L_n(U))^2 \to L_n(U)$ satisfies $\mathcal{I}2$, $\mathcal{I}5$, $\mathcal{I}13$ and verifies the right-continuity w.r.t. the second variable, then $\mathcal{T}_{\mathcal{I}}$ defined by Eq.(66) is a left-continuous n-DT. Moreover $\mathcal{I} = \mathcal{I}_{\mathcal{T}_{\mathcal{I}}}$, meaning that*

$$\mathcal{I}(\mathbf{x}, \mathbf{y}) = \max\{\mathbf{t} \in L_n(U) \colon \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{t}) \leq \mathbf{y}\}. \tag{69}$$

Proof: Since $\mathcal{I}$ satisfies $\mathcal{I}5$ and $\mathcal{I}13$, by Proposition 5.1.1, it satisfies $\mathcal{I}0(a)$, $\mathcal{I}0(b)$, $\mathcal{I}1$, $\mathcal{I}3$ and $\mathcal{I}14$. In particular, $\mathcal{I}$ satisfies $\mathcal{I}2$ and then it is an n-DI. For each $\mathbf{x}, \mathbf{y} \in L_n(U)$ the following results are verified.

($\mathcal{T}1$) For $\mathbf{x} \in L_n(U)$, by $\mathcal{I}3$, we get the following:

$$\mathcal{T}_{\mathcal{I}}(\mathbf{x}, /1/) = \mathcal{T}_{\mathcal{I}}(/1/, \mathbf{x}) = \inf\{\mathbf{t} \in L_n(U) \colon \mathcal{I}(/1/, \mathbf{t}) \geq \mathbf{x}\} = \inf\{\mathbf{t} \in L_n(U) \colon \mathbf{t} \geq \mathbf{x}\} = \mathbf{x}.$$

($\mathcal{T}2$) By $\mathcal{I}5$, for each $\mathbf{t} \in L_n(U)$, $\mathcal{I}(\mathbf{y}, \mathcal{I}(\mathbf{x}, \mathbf{t})) = 1 \Leftrightarrow \mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y}, \mathbf{t})) = 1$. And, by $\mathcal{I}13$, it means that $\mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y} \Leftrightarrow \mathcal{I}(\mathbf{y}, \mathbf{t}) \geq \mathbf{x}$. Then, the following holds:

$$\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) = \min\{\mathbf{t} \in L_n(U) : \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}\} = \min\{\mathbf{t} \in L_n(U) : \mathcal{I}(\mathbf{y}, \mathbf{t}) \geq \mathbf{x}\} = \mathcal{T}_{\mathcal{I}}(\mathbf{y}, \mathbf{x}).$$

($\mathcal{T}3$) Let $\mathbf{x}, \mathbf{y}, \mathbf{z} \in L_n(U)$ and $\mathbf{t} \in L_n(U)$. From ($\mathcal{RP}$) and $\mathcal{I}5$:

$$\begin{aligned}
\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathcal{T}_{\mathcal{I}}(\mathbf{y}, \mathbf{z})) &= \inf\{\mathbf{t} \in L_n(U) : \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathcal{T}_{\mathcal{I}}(\mathbf{y}, \mathbf{z})\} \\
&= \inf\{\mathbf{t} \in L_n(U) : \mathcal{I}(\mathbf{y}, \mathcal{I}(\mathbf{x}, \mathbf{t})) \geq \mathbf{z}\} \\
&= \inf\{\mathbf{t} \in L_n(U) : \mathcal{I}(\mathbf{x}, \mathcal{I}(\mathbf{y}, \mathbf{t})) \geq \mathbf{z}\} \\
&= \inf\{\mathbf{t} \in L_n(U) : \mathcal{I}(\mathbf{y}, \mathbf{t}) \geq \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{z})\} = \mathcal{T}_{\mathcal{I}}(\mathbf{y}, \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{z})).
\end{aligned}$$

($\mathcal{T}4$) For $\mathbf{x}, \mathbf{y}_1, \mathbf{y}_2 \in L_n(U)$ and $\mathbf{y}_1 \leq \mathbf{y}_2$, we have that

$$\{\mathbf{t} \in L_n(U) \colon \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}_1\} \supset \{\mathbf{t} \in L_n(U) \colon \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathbf{y}_2\}.$$

Then, it implies that $\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}_1) \leq \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}_2)$.

So, we conclude that $\mathcal{T}$ is a n-DT.

Moreover, let us assume that $\mathcal{T}_{\mathcal{I}}$ is not left-continuous w.r.t the second variable at some point $(\mathbf{x}_0, \mathbf{y}_0) \in (L_n(U))^2$. Since as an n-DT $\mathcal{T}_{\mathcal{I}}$ is increasing, there exist $\mathbf{a}, \mathbf{b} \in L_n(U)$ such that $\mathbf{a} \leq \mathbf{b}$ and $\mathcal{T}_{\mathcal{I}}(\mathbf{x}_0, \mathbf{y}) \leq \mathbf{a}$, for all $\mathbf{y} < \mathbf{y}_0, \mathcal{T}_{\mathcal{I}}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{b}$. By $\mathcal{RP}$, we have $\mathcal{I}(\mathbf{x}_0, \mathbf{a}) \geq \mathbf{y}$, for all $\mathbf{y} \leq \mathbf{y}_0$. In the limit given as $\lim_{i\to\infty} \mathbf{y}_i = \mathbf{y}_0$, we get $\mathcal{I}(\mathbf{x}_0, \mathbf{a}) \geq \mathbf{y}_0$. Applying $\mathcal{RP}$, $\mathbf{b} = \mathcal{T}(\mathbf{x}_0, \mathbf{y}_0) \leq \mathbf{a}$, which is a contradiction to $\mathbf{a} < \mathbf{b}$. Then, $\mathcal{T}_{\mathcal{I}}$ is a left-continuous n-DT.

($\mathcal{RC}$): And, consider $\mathcal{T}_{\mathcal{I}}$ as a binary function which does not verify the right-continuity property w.r.t. the second variable in some point $(\mathbf{x}_0, \mathbf{y}_0) \in L_n(U) \times L_n(U)$. Thus, there exists an decreasing chain $(\mathbf{y}_i)_{i\in\mathbb{N}}$ such that $\lim_{i\to\infty} \mathbf{y}_i = \mathbf{y}_0$ and $\mathbf{a} = \lim_{i\to\infty} \mathcal{T}_{\mathcal{I}}(\mathbf{x}_0, \mathbf{y}_i) \neq \mathcal{T}_{\mathcal{I}}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{b}$. So, because for each $i \in \mathbb{N}$, $\mathbf{y}_i \geq \mathbf{y}_0$ and $\mathcal{T}_{\mathcal{I}}$ satisfies $\mathcal{T}3$, then $\mathbf{a} > \mathbf{b}$. Since, for each $i \in \mathbb{N}$, $\mathcal{T}_{\mathcal{I}}(\mathbf{x}_0, \mathbf{y}_i) \leq \mathbf{a}$ then, by Property $\mathcal{RP}$, we obtain that $\mathcal{I}(\mathbf{x}_0, \mathbf{a}) \geq \mathbf{y}_i$ and therefore $\lim_{i\to\infty} \mathcal{I}(\mathbf{x}_0, \mathbf{y}_i) \geq \lim_{i\to\infty} \mathbf{y}_i = \mathbf{y}_0$. Hence, since $\mathcal{I}$ is right-continuous, we have $\mathcal{I}(\mathbf{x}_0, \mathbf{a}) \geq \mathbf{y}_0$. Therefore, from $\mathcal{RP}$, $\mathbf{a} \leq \mathcal{T}_{\mathcal{I}}(\mathbf{x}_0, \mathbf{y}_0) = \mathbf{b}$, which is a contradiction with hypothesis $\mathbf{a} > \mathbf{b}$. Concluding, $\mathcal{T}_{\mathcal{I}}$ is a right-continuous function w.r.t. the second variable.

Now, let $\mathcal{I}_{\mathcal{T}_{\mathcal{I}}}$ be an n-RDI. For each $\mathbf{x}, \mathbf{y} \in L_n(U)$, by Proposition 6.4.1, the following is verified:

$$\mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathcal{I}(\mathbf{x}, \mathbf{y})) = \min\{t \in L_n(U) \colon \mathcal{I}(\mathbf{x}, \mathbf{t}) \geq \mathcal{I}(\mathbf{x}, \mathbf{y})\} \leq \mathbf{y} \Rightarrow \mathcal{I}(\mathbf{x}, \mathbf{y}) \in \{\mathbf{t} \in L_n(U) \colon \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{t}) \leq \mathbf{y}\}.$$

Then, $\mathcal{I}(\mathbf{x}, \mathbf{y}) \leq \mathcal{I}_{\mathcal{T}_{\mathcal{I}}}(\mathbf{x}, \mathbf{y})$. By $\mathcal{I}(\mathbf{x}, \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{z}) \geq \mathbf{z}$, for $\mathbf{z} \in L_n(U)$, if $\mathbf{z} = \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y})$. In addition, since $\mathcal{T}_{\mathcal{I}}$ is left-continuous, from Proposition 6.2.1 we know that the pair ($\mathcal{T}_{\mathcal{I}}, \mathcal{I}_{\mathcal{T}_{\mathcal{I}}}$) satisfies $\mathcal{RP}$ and by $\mathcal{I}2$, we have that:

$$\mathcal{I}_{\mathcal{T}_{\mathcal{I}}}(\mathbf{x}, \mathbf{y}) \leq \mathcal{I}(\mathbf{x}, \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathcal{I}_{\mathcal{T}_{\mathcal{I}}}(\mathbf{x}, \mathbf{y}))) \leq \mathcal{I}(\mathbf{x}, \mathbf{y}).$$

So, $\mathcal{I}_{\mathcal{T}_{\mathcal{I}}}(\mathbf{x}, \mathbf{y}) = \mathcal{I}(\mathbf{x}, \mathbf{y})$. And, Theorem 6.4.1 holds. □

We also have the following connection between left-continuous n-DT and n-RDI generated from them.

Lemma 6.4.1 *$\mathcal{T} = \mathcal{T}_{\mathcal{I}_{\mathcal{T}}}$ if $\mathcal{T}$ is a left-continuous n-DT.*

Proof: From Proposition 6.2.1, the function $\mathcal{I}_{\mathcal{T}} \in \mathcal{I}(L_n(U))$ satisfies $\mathcal{I}5$, $\mathcal{I}13$ and is right-continuous w.r.t. the second variable. By Theorem 6.4.1, the function $\mathcal{T}_{\mathcal{I}_{\mathcal{T}}}$ is a left-continuous n-DT. Since $\mathcal{T}$ is left-continuous and by Proposition 6.2.1 the pair ($\mathcal{T}, \mathcal{I}_{\mathcal{T}}$) satisfies $\mathcal{RP}$, then for $\mathbf{x}, \mathbf{y} \in L_n(U)$ it holds that:

$$\mathcal{I}_{\mathcal{T}}(\mathbf{x}, \mathcal{T}(\mathbf{x}, \mathbf{y})) = \max\{\mathbf{t} \in L_n(U) \colon \mathcal{T}(\mathbf{x}, \mathbf{t}) \leq \mathcal{T}(\mathbf{x}, \mathbf{y})\} \geq \mathbf{y},$$

i.e., $\mathcal{T}(\mathbf{x},\mathbf{y}) \in \{\mathbf{t} \in L_n(U) \colon \mathcal{I}_{\mathcal{T}}(\mathbf{x},\mathbf{t}) \geq \mathbf{y}\}$ and hence

$$\mathcal{T}(\mathbf{x},\mathbf{y}) \geq \mathcal{T}_{\mathcal{I}_{\mathcal{T}}}(\mathbf{x},\mathbf{y}). \tag{70}$$

Conversely, since $\mathcal{T}$ is left-continuous, from Proposition 6.2.1 it follows that $\mathcal{T}(\mathbf{x},\mathcal{I}_{\mathcal{T}}(\mathbf{x},\mathbf{z})) \leq \mathbf{z}$, for any $\mathbf{z} \in L_n(U)$. If $\mathbf{z} = \mathcal{T}_{\mathcal{I}_{\mathcal{T}}}(\mathbf{x},\mathbf{y})$, then $\mathcal{T}_{\mathcal{I}_{\mathcal{T}}}(\mathbf{x},\mathbf{y}) \geq \mathcal{T}(\mathbf{x},\mathcal{I}_{\mathcal{T}}(\mathbf{x},\mathcal{T}_{\mathcal{I}_{\mathcal{T}}}(\mathbf{x},\mathbf{y})))$. Further, since $\mathcal{I}_{\mathcal{T}}$ is right-continuous, from Proposition 6.4.1 we also get $\mathcal{I}_{\mathcal{T}}(\mathbf{x},\mathcal{T}_{\mathcal{I}_{\mathcal{T}}}(\mathbf{x},\mathbf{y})) \geq \mathbf{y}$. Making use of such inequality in Eq.(70), by monotonicity we have

$$\mathcal{T}_{\mathcal{I}_{\mathcal{T}}}(\mathbf{x},\mathbf{y}) \geq \mathcal{T}(\mathbf{x},\mathbf{y}). \tag{71}$$

From Eqs.(70) and (71), we get $\mathcal{T}_{\mathcal{I}_{\mathcal{T}}}(\mathbf{x},\mathbf{y}) = \mathcal{T}(\mathbf{x},\mathbf{y})$, $\mathbf{x},\mathbf{y} \in L_n(U)$. So, Lemma 6.4.1 is verified. □

Other results related to the characterization of n-DRI generated from left-continuous n-DT are shown below.

Theorem 6.4.2 *For a function $\mathcal{I} : (L_n(U))^2 \to L_n(U)$ the following statements are equivalent:*
***1**. $\mathcal{I}$ is an n-DRI generated from a left-continuous n-DT.*
***2**. $\mathcal{I}$ satisfies $\mathcal{I}2, \mathcal{I}5, \mathcal{I}13$ and it is right-continuous w.r.t the second variable.*
Moreover, the representation of n-DRI, up to a left-continuous n-DT, is unique in this case.

Proof: Consider function $\mathcal{I} : (L_n(U))^2 \to L_n(U)$.

$(1) \Rightarrow (2)$ Let $\mathcal{I}$ be an n-DRI generated from a left-continuous n-DT $\mathcal{T}_{\mathcal{I}}$. From Proposition 6.2.1 it satisfies $\mathcal{I}2, \mathcal{I}5, \mathcal{I}13$ and it is right-continuous w.r.t the second variable.

$(2) \Rightarrow (1)$ Let $\mathcal{I} : (L_n(U))^2 \to L_n(U)$ be a function satisfying $\mathcal{I}2, \mathcal{I}5, \mathcal{I}13$ and it is right-continuous w.r.t. the second variable. By Theorem 6.4.1, we get that $\mathcal{I} = \mathcal{I}_{\mathcal{T}_{\mathcal{I}}}$ where $\mathcal{T}_{\mathcal{I}}$ defined by Eq.(66) is a left-continuous n-DT. Hence $\mathcal{I}$ is an n-DRI generated from a left-continuous n-DT $\mathcal{T}_{\mathcal{I}}$. And, the uniqueness of the representation of an n-DRI up to a left-continuous n-DT follows from Lemma 6.4.1.

Concluding, it is shown that Theorem 6.4.2 is verified. □

The next corollary follows from Theorem 6.4.2 characterizing left-continuous t-norms on $L_n(U)$.

Corollary 6.4.1 *For a function $\mathcal{T} : (L_n(U))^2 \to L_n(U)$, the following statements are equivalent:*
***1**. $\mathcal{T}$ is a left-continuous n-DT.*
***2**. There exists $\mathcal{I} \in \mathcal{I}(L_n(U))$, which satisfies $\mathcal{I}5$, $\mathcal{I}13$ and it is also right-continuous w.r.t. the second variable, such that $\mathcal{T}$ is given by Eq.(66).*

Proof: Straightforward from Theorem 6.4.2. □

Lemma 6.4.2 *If $\mathcal{I} : (L_n(U))^2 \to L_n(U)$ is a continuous function except at the point $(/0/, /0/)$ satisfying $\mathcal{I}2$, $\mathcal{I}5, \mathcal{I}11$ and $\mathcal{I}(\mathbf{x}, /0/) = N_{D1}(\mathbf{x})$, then the function $\mathcal{T}_{\mathcal{I}}$ defined by Eq.(65) is a continuous n-DT.*

Proof: Since $\mathcal{I}$ is right continuous w.r.t.the second variable, from Theorem 6.4.1, $\mathcal{T}_{\mathcal{I}}$ is a left continuous n-DT. In addition, Eq.(65) implies Eq.(66) meaning that $(\mathcal{T}_{\mathcal{I}}, \mathcal{I})$ is an adjoint pair, i.e., $\mathcal{I}$ and $\mathcal{T}_{\mathcal{I}}$ satisfy property $\mathcal{RP}$. Thus, in order to prove that n-DT $\mathcal{T}_{\mathcal{I}}$ is a continuous positive Archimedean function, suppose the contrary, that $\mathcal{T}_{\mathcal{I}}$ is not right-continuous. Then there exist $\mathbf{x}, \mathbf{y} \in L_n(U)$ and $\mathbf{z}, \mathbf{z}' \in L_n(U)$ such that

$$\mathbf{z} = \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y}) < \mathbf{z}' < \lim_{h \to 0^+} \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y} + \mathbf{h}). \tag{72}$$

From $\mathcal{RP}$ property, we have $\mathcal{I}(\mathbf{x}, \mathbf{z}) \geq \mathbf{y}$ and by $\mathcal{I}2$, we have $\mathcal{I}(\mathbf{x}, \mathbf{z}') \geq \mathbf{y}$. Moreover, by $\mathcal{RP}$, we also have that $\mathbf{z}' \leq \mathcal{T}_{\mathcal{I}}(\mathbf{x}, \mathbf{y})$, which is a contradiction. Thus, $\mathcal{T}_{\mathcal{I}}$ is right-continuous and hence, it is also continuous. Therefore, Lemma 6.4.2 is verified. □

6.5 Obtaining n-DRI from n-DA operators

This section extends main results in (LIU; WANG, 2006) from interval-valued fuzzy set theory to n-dimensional simplex. In particular, the class of n-DRI is constructed based on binary n-DA aggregation operators, given as the minimum operator and left-continuous t-norms.

Proposition 6.5.1 *Let $T_1, \ldots, T_n : U^2 \to U$ be left-continuous t-norms such that $T_1 \leq \ldots \leq T_n$. Then, for all $\mathbf{x} = (x_1, \ldots., x_n)$, $\mathbf{y} = (y_1, \ldots, y_n) \in L_n(U)$ the function $\mathcal{T}_{T_1,\ldots,T_n} : (L_n(U))^2 \to L_n(U)$ given as*

$$\mathcal{T}_{T_1 \ldots T_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=1}^{n} T_1(x_{n-i+1}, y_i), \min_{i=2}^{n} T_2(x_{n-i+2}, y_i), \ldots, T_n(x_n, y_n) \right), \tag{73}$$

and shortly expressed as

$$\mathcal{T}_{T_1 \ldots T_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=k}^{n} T_k(x_{n-i+k}, y_i) \right)_{k \in \mathbb{N}_n^*} \tag{74}$$

verifies $\mathcal{T}1$, $\mathcal{T}2$, $\mathcal{T}3$, $\mathcal{LC}$ and $\mathcal{RP}$ properties on $L_n(U)$.

Proof: Let $T_1, \ldots, T_n : U^2 \to U$ be left-continuous t-norms such that $T_1 \leq ... \leq T_n$. For all $\mathbf{x}, \mathbf{y} \in L_n(U)$ the following holds:

$$(\mathcal{T}1)\mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, /1/) = \left(\min_{i=k}^{n} T_k(x_{n-i+k}, 1)\right)_{k\in\mathbb{N}_n^*} = \left(\min_{i=k}^{n} x_{n-i+k}\right)_{k\in\mathbb{N}_n^*}$$
$$= (\min(x_n, x_{n-1}, \ldots, x_1), \min(x_n, x_{n-1}, \ldots, x_2), \ldots, x_n) = (x_1, x_2, \ldots, x_n) = \mathbf{x}.$$
$$(\mathcal{T}2)\mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=k}^{n} T_k(x_{n-i+k}, y_i)\right)_{k\in\mathbb{N}_n^*} = \left(\min_{i=k}^{n} T_k(y_i, x_{n-i+k})\right)_{k\in\mathbb{N}_n^*} = \left(\min_{i=k}^{n} T_k(y_{n-i+k}, x_i)\right)_{k\in N_n^*}.$$

Then, we have that $\mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathbf{y}) = \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{y}, \mathbf{x})$.

$$(\mathcal{T}3)\mathbf{x} \leq \mathbf{x}' \Rightarrow \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathbf{y})) = \left(\min_{i=k}^{n} T_k(x_{n-i+k}, y_i)\right)_{k\in\mathbb{N}_n^*} \leq \left(\min_{i=k}^{n} T_k(x'_{n-i+k}, y_i)\right)_{k\in\mathbb{N}_n^*}$$
$$= \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}', \mathbf{y}).$$

($\mathcal{LC}$) Let $\mathbf{x}^l$ be a non-decreasing sequence in $L_n(U)$. Then, the following holds:

$$\lim_{l\to\infty} \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}^l, \mathbf{y}) = \lim_{l\to\infty} \left(\min_{i=k}^{n} T_k(x^l_{n+k-i}, y_i)\right)_{k\in\mathbb{N}_n^*}$$
$$= \lim_{k\to\infty} \left(\min_{i=1}^{n} T_1(x^l_{n+1-i}, y_i), \min_{i=2}^{n} T_2(x^l_{n+2-i}, y_i), \ldots, T_n(x^l_n, y_n)\right)$$
$$= \left(\min_{i=1}^{n} T_1(\lim_{l\to\infty} x^l_{n+1-i}, y_i), \min_{i=2}^{n} T_2(\lim_{l\to\infty} x^l_{n+2-i}, y_i), \ldots, T_n(\lim_{l\to\infty} x^l_n, y_n)\right)$$
$$= \left(\min_{i=k}^{n} T_k(\lim_{l\to\infty} x^l_{n+k-i}, y_i)\right)_{k\in\mathbb{N}_n^*} = \mathcal{T}_{T_1,\ldots,T_n}(\lim_{k\to\infty} \mathbf{x}^l, \mathbf{y}).$$

Since, $\mathcal{T}_{T_1,\ldots,T_n}$ satisfies $\mathcal{LC}$ and the proof of Theorem 6.2.1 ($i \to ii$) does not make use of $\mathcal{T}2$, then this same proof also proves ($\mathcal{RP}$) for $\mathcal{T}_{T_1,\ldots,T_n}$ and $\mathcal{I}_{\mathcal{T}T_1,\ldots,T_n}$.

Concluding, Proposition 6.5.1 is verified. □

In the next Corollary, it is shown that the operator $\mathcal{T}_{T_1,\ldots,T_n}$ forms an adjoint pair with their residuum operator, in spite of not being necessarily a n-dimensional t-norm.

Corollary 6.5.1 *Let $T_1, \ldots, T_n : U^2 \to U$ be left-continuous t-norms such that $T_1 \leq \ldots \leq T_n$. Then, $(\mathcal{T}_{T_1,\ldots,T_n}, \mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}})$ is an adjoint pair.*

Proof: Straightforward from Propositions 6.4.1 and 6.5.1. □

The characterization of an $\mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}$ operator is presented in the following theorem.

Theorem 6.5.1 *Let $T_1, \ldots, T_n : U^2 \to U$ be left-continuous t-norms on U such that $T_1 \leq \ldots \leq T_n$ and $I_1, \ldots, I_n$ be their corresponding residual implications. Then the operator $\mathcal{I}_{I_1,\ldots,I_n} = \mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}$ where $\mathcal{I}_{I_1,\ldots,I_n} : (L_n(U))^2 \to L_n(U)$ defined as*

$$\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}, \mathbf{y}) = \left(\min_{i=1}^{n} I_i(x_i, y_i), \min_{i=2}^{n} I_i(x_i, y_i), \ldots, I_n(x_n, y_n)\right) = \left(\min_{i=k}^{n} I_i(x_i, y_i)\right)_{k\in\mathbb{N}_n^*}. \quad (75)$$

Proof: Let $T_1, \ldots, T_n : U^2 \to U$ be left-continuous t-norms on U such that $T_1 \leq \ldots \leq T_n$ and $I_1, \ldots, I_n$ be their corresponding residual implications, for all $\mathbf{x}, \mathbf{y} \in L_n(U)$ we have the next results:

$$
\begin{aligned}
\mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y})) \leq \mathbf{y} &\Leftrightarrow \min_{i=k}^{n} T_k(x_{n+k-i}, \min_{j=k}^{n} I_j(x_j, y_j)) \leq y_k, \forall k \in \{1, \ldots, n\} \\
&\Leftrightarrow T_k(x_k, \min_{j=k}^{n} I_j(x_j, y_j)) \leq y_k, \forall k \in \{1, \ldots, n\} \\
&\Leftrightarrow \min_{j=k}^{n} I_j(x_j, y_j) \leq I_k(x_k, y_k), \forall k \in \{1, \ldots, n\} (\text{ by } \mathcal{RP}).
\end{aligned}
$$

So, $\mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x}, \mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y})) \leq \mathbf{y}$, therefore, $\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y}) \in \{\mathbf{z} \in L_n(U) : \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x},\mathbf{z}) \leq \mathbf{y}\}$.
Consequently, it holds that:
(i) $\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y}) \leq \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x},\mathbf{z}) \leq \mathbf{y}\} = \mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}(\mathbf{x},\mathbf{y})$.

Now, consider $\mathbf{z} \in L_n(U)$ such that $\mathcal{T}_{T_1,\ \ldots,\ T_n}(\mathbf{x},\ \mathbf{z}) \leq \mathbf{y}$. Then, for each $k \in \{1, \ldots, n\}$, $\min_{i=k}^{n} T_k(x_{n+k-i}, z_i) \leq y_k$. So, there exists $i \geq k$ such that $T_k(x_{n+k-i}, z_i) \leq y_k$ and therefore, $T_k(x_k, z_k) \leq y_k$. Hence, by $\mathcal{RP}$, $z_k \leq I_k(x_k, y_k)$ for each $k \in \{1, \ldots, n\}$ and therefore $z_k \leq \min_{i=k}^{n} I_i(x_i, y_i)$. This is implied in the following equation:

$$
\mathbf{z} \leq \left(\min_{i=1}^{n} I_i(x_i, y_i), \min_{i=2}^{n} I_i(x_i, y_i), \ldots, I_n(x_n, y_n) \right) = \mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y}).
$$

Consequently, one can also conclude the second one:
(ii) $\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y}) \geq \sup\{\mathbf{z} \in L_n(U) : \mathcal{T}_{T_1,\ldots,T_n}(\mathbf{x},\mathbf{z}) \leq \mathbf{y}\} = \mathcal{I}_{\mathcal{T}_{T_1,\ldots,T_n}}(\mathbf{x},\mathbf{y})$.

Thus, from (i) e (ii), Theorem 6.5.1 follows. □

Corollary 6.5.2 *Let $T_1, \ldots, T_n : U^2 \to U$ be left-continuous t-norms on U such that $T_1 \leq \ldots \leq T_n$ and $I_1, \ldots, I_n$ be their corresponding residual implications. Then $(\mathcal{I}_{I_1,\ldots,I_n}, \mathcal{T}_{T_1 \ldots T_n})$ is an adjoint pair on $L_n(U)$.*

Proof: Straightforward from Corollary 6.5.1 and Theorem 6.5.1. □

6.5.1 Preserving main implication properties from the action of $\mathcal{I}_{I_1,\ldots,I_n}$-operator

The following proposition investigates the conditions under which the main properties of fuzzy implications are preserved by the action of $\mathcal{I}_{I_1,\ldots,I_n}$-operator on $L_n(U)$.

Proposition 6.5.2 *Let $I_1, \ldots, I_n : U^2 \to U$ be implications such that $I_n \leq \ldots \leq I_1$ and $k \in \{0, 1, 2, 3, 5, 12, 13, 14\}$. The function $\mathcal{I}_{I_1,\ldots,I_n} : (L_n(U))^n \to L_n(U)$ given by*

$$
\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y}) = \left(\min_{i=1}^{n} I_i(x_i, y_i), \min_{i=2}^{n} I_i(x_i, y_i), \ldots, I_n(x_n, y_n) \right), \tag{76}
$$

and shortly expressed as

$$\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y}) = \left(\min_{i=k}^{n} I_i(x_i,y_i)\right)_{k\in\mathbb{N}_n^*}, \tag{77}$$

verifies $\mathcal{I}_k$ property.

Proof: $I_1,\ldots,I_n : U^2 \to U$ be implications such that $I_n \leq ... \leq I_1$. The following results are verified:

($\mathcal{I}0$): Firstly, the boundary conditions hold:

$$\begin{aligned}
\mathcal{I}_{I_1,\ldots,I_n}(/0/,/0/) &= \left(\min_{i=k}^{n} I_i(0,0)\right)_{k\in\mathbb{N}_n} = /1/;\\
\mathcal{I}_{I_1,\ldots,I_n}(/0/,/1/) &= \left(\min_{i=k}^{n} I_i(0,1)\right)_{k\in\mathbb{N}_n} = /1/;\\
\mathcal{I}_{I_1,\ldots,I_n}(/1/,/1/) &= \left(\min_{i=k}^{n} I_i(1,1)\right)_{k\in\mathbb{N}_n} = /1/;\\
\mathcal{I}_{I_1,\ldots,I_n}(/1/,/0/) &= \left(\min_{i=k}^{n} I_i(1,0)\right)_{k\in\mathbb{N}_n} = /0/.
\end{aligned}$$

($\mathcal{I}1$): If $\mathbf{x} \leq \mathbf{x}'$ then the following holds:

$$\begin{aligned}
\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y}) &= \left(\min_{i=k}^{n} I_{I_i}(x_i,y_i)\right)_{k\in\mathbb{N}_n} = \left(\min_{i=1}^{n} I_1(x_i,y_i), \min_{i=2}^{n} I_i(x_i,y_i),\ldots,I_n(x_n,y_n)\right)\\
&\geq \left(\min_{i=1}^{n} I_1(x_i',y_i), \min_{i=2}^{n} I_n(x_i',y_i),\ldots,I_n(x_n',y_n)\right)\\
&= \left(\min_{i=k}^{n} I_{T_i}(x_i',y_i)\right)_{k\in\mathbb{N}_n} = \mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x}',\mathbf{y}).
\end{aligned}$$

($\mathcal{I}2$): If $\mathbf{y} \leq \mathbf{y}'$ then the next results hold:

$$\begin{aligned}
\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y}) &= \left(\min_{i=k}^{n} I_{I_i}(x_i,y_i)\right)_{k\in\mathbb{N}_n} = \left(\min_{i=1}^{n} I_1(x_i,y_i), \min_{i=2}^{n} I_i(x_i,y_i),\ldots,I_n(x_n,y_n)\right)\\
&\leq \left(\min_{i=1}^{n} I_1(x_i,y_i'), \min_{i=2}^{n} I_i(x_i,y_i'),\ldots,I_n(x_n,y_n')\right)\\
&= \left(\min_{i=k}^{n} I_{T_i}(x_i,y_i')\right)_{k\in\mathbb{N}_n} = \mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y}').
\end{aligned}$$

$$\begin{aligned}
(\mathcal{I}3) : \mathcal{I}_{I_1,\ldots,I_n}(/1/,\mathbf{y}) &= \left(\min_{i=k}^{n} I_i(1,y_i)\right)_{k\in\mathbb{N}_n} = \left(\min_{i=1}^{n} I_1(1,y_i), \min_{i=2}^{n} I_i(1,y_i),\ldots,I_i(1,y_n)\right)\\
&= \left(\min_{i=1}^{n} y_i, \min_{i=2}^{n} y_i,\ldots,y_n\right) = (y_1,y_2,\ldots,y_n) = \mathbf{y}.
\end{aligned}$$

$$(\mathcal{I}5):\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},(\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{y},\mathbf{z}))=\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\left(\min_{i=k}^{n} I_i(y_i,z_i)\right)_{k\in\mathbb{N}_n}$$
$$=\left(\min_{i=l}^{n} I_i\left(x_i,\left(\min_{i=k}^{n} I_i(y_i,z_i)\right)_{k\in\mathbb{N}_n}\right)\right)_{l\in\mathbb{N}_n}$$
$$=\left(\min_{i=l}^{n} I_i\left(\left(\min_{i=k}^{n} I_i(x_i,y_i)\right)_{k\in\mathbb{N}_n},z_i\right)\right)_{l\in\mathbb{N}_n}=\mathcal{I}_{I_1,\ldots,I_n}(\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y}),\mathbf{z}).$$

$$(\mathcal{I}12):\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},/0/)=\left(\min_{i=k}^{n} I_i(x_i,0)\right)_{k\in\mathbb{N}_n^*}=\left(\min_{i=1}^{n} I_i(x_i,0),\min_{i=2}^{n} I_i(x_i,0),\ldots,I_i(x_n,0)\right)$$
$$=(\min(I_1(x_1,0),I_2(x_2,0),\ldots,I_n(x_n,0)),\min(I_2(x_2,0),\ldots,I_n(x_n,0)),\ldots,I_n(x_n,0))$$
$$=(I_n(x_n,0),I_n(x_n,0),\ldots I_n(x_n,0))$$
$$=(N_n(x_n),N_n(x_n)\ldots,N_n(x_n))=\mathcal{N}_{N_n}(/x_n/).$$

$$(\mathcal{I}13):\mathcal{I}_{T_1,\ldots,T_n}(\mathbf{x},\mathbf{y})=/1/\Leftrightarrow\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{y})=/1/$$
$$\Leftrightarrow\left(\min_{i=1}^{n} I_{T_i}(x_i,y_i),\min_{i=2}^{n} I_{T_i}(x_i,y_i),\ldots,I_{T_n}(x_n,y_n)\right)=/1/$$
$$\Leftrightarrow\min_{i=1}^{n} I_{T_i}(x_i,y_i)=1 \text{ or } \min_{i=2}^{n} I_{T_i}(x_i,y_i)=1 \text{ or } \ldots \text{ or } I_{T_n}(x_n,y_n)=1$$
$$\Leftrightarrow x_n\le y_n \text{ and } x_{n-1}\le y_{n-1} \text{ and } \ldots \text{ and } x_1\le y_1\Leftrightarrow\mathbf{x}\le\mathbf{y}.$$

$$(\mathcal{I}14):\mathcal{I}_{I_1,\ldots,I_n}(\mathbf{x},\mathbf{x})=\left(\min_{i=k}^{n} I_{I_i}(x_i,x_i)\right)_{k\in\mathbb{N}_n}=\left(\min_{i=1}^{n} I_1(x_i,x_i),\min_{i=2}^{n} I_i(x_i,x_i),\ldots,I_n(x_n,x_n)\right)=/1/.$$

Therefore, Proposition 6.5.2 is verified. □

In the next theorem, the pair of functions $\left(\mathcal{T}_{T_1\ldots T_n},\mathcal{I}_{\mathcal{A}_{T_1\ldots T_n}}\right)$ is characterized as an adjoint pair, meaning that $\mathcal{T}_{T_1\ldots T_n}$ and $\mathcal{I}_{\mathcal{A}_{T_1\ldots T_n}}$ verify the Residuation Principle.

Theorem 6.5.2 *Let $T_1,\ldots,T_n : U^2 \to U$ be left-continuous t-norms such that $T_1 \le \ldots \le T_n$ and $\mathcal{A}_{T_1,\ldots,T_n} : (L_n(U))^2 \to L_n(U)$ be n-DT verifying the conditions of Proposition 6.5.1. When $I_{T_i} : U^2 \to U$ is the residual implication generated by T_i, for $i \in \mathbb{N}_n$. The residual implication generated by $\mathcal{A}_{T_1\ldots T_n}$ is the function $\mathcal{I}_{T_1,\ldots,T_n} : (L_n(U))^n \to L_n(U)$ verifying the condition of Proposition 6.5.2.*

Proof: Since the operators $\min, T_1\ldots T_n : (L_n(U))^2 \to L_n(U)$ are left-continuous binary functions, they satisfy the residuation principle meaning that the following holds:

$$T_i(x_i,t_i)\le y_i\Leftrightarrow I_{T_i}(x_i,y_i)\ge t_i,\forall i\in\mathbb{N}_n.$$

Then, one can verify the next results:

$$
\begin{aligned}
\mathcal{T}_{T_1\dots T_n}(\mathbf{x},\mathbf{y}) \le \mathbf{t} &\Leftrightarrow \left(\min_{i=k}^{n} T_k(x_{n-i+k}, y_i)\right)_{k\in\mathbb{N}_n^*} \le \mathbf{t}\\
&\Leftrightarrow \left(\min_{i=1}^{n} T_1(x_{n-i+1}, y_i), \min_{i=2}^{n} T_2(x_{n-i+2}, y_i), \dots, T_n(x_n, y_n)\right) \le \mathbf{t}\\
&\Leftrightarrow \min_{i=1}^{n} T_1(x_{n-i+1}, y_i) \le t_1, \min_{i=2}^{n} T_2(x_{n-i+2}, y_i) \le t_2, \dots, T_n(x_n, y_n) \le t_n\\
&\Leftrightarrow \min_{i=1}^{n} I_{T_1}(x_{n-i+1}, t_1) \ge y_1, \min_{i=2}^{n} I_{T_2}(x_{n-i+2}, t_2) \ge y_2, \dots, I_{T_n}(x_n, t_n) \ge y_n\\
&\Leftrightarrow \left(\min_{i=1}^{n} I_{T_i}(x_i, t_i), \min_{i=2}^{n} I_{T_i}(x_i, t_i), \dots, I_{T_n}(x_n, t_n)\right) \ge \mathbf{y}\\
&\Leftrightarrow \left(\min_{i=j}^{n} I_{T_i}(x_i, t_i)\right)_{j\in\mathbb{N}_n} \ge \mathbf{y}.
\end{aligned}
$$

Therefore, Theorem 6.5.2 is verified. □

6.6 Solving a MCDM problem in CIM-application

This section extents the application described in (WEN; ZHAO; XU, 2018, Example 1) from HFS to n-DFS, in order to solve the MCDM considering multiple alternatives in the selection of CIM (Computer-Integrated Manufacturing) software.

6.6.1 Describing the CIM-MCDM problem

In order to help the user in the selection of seven kinds of CIM software systems available in the market nowadays, a data processing company aims to clarify differences of such systems (CHEN; XU; XIA, 2013). The evaluations expressed by n-DFS are shown in Table 3. In this case study, $A = \{A_1, A_2, \dots, A_7\}$ ($n_2 = 7$) is the set of CIM software alternatives and X is the set of 4 attributes related to functionality (x_1), usability (x_2), portability (x_3) and maturity (x_4) ($n_1 = 4$). See Table 3, related to matrix $[D]_{7\times 4} = (\mathbf{x}_{ki})_{k=1\dots7, i=1\dots4}$ whose elements are given as an 3-dimensional interval $\mathbf{x}_{ij}$, containing selected opinions of 3 decision makers and providing their evaluations with values between 0 and 1 for all alternatives A_i w.r.t. each attribute.

Table 3 – Information related to the n-dimensional interval components.

D-matrix	$\mathbf{x_1}$	$\mathbf{x_2}$	$\mathbf{x_3}$	$\mathbf{x_4}$
A1	(0.80, 0.85, 0.95)	(0.70, 0.75, 0.80)	(0.65, 0.65, 0.80)	(0.30, 0.30, 0.35)
A2	(0.85, 0.85, 0.90)	(0.60, 0.70, 0.80)	(0.20, 0.20, 0.20)	(0.15, 0.15, 0.15)
A3	(0.20, 0.30, 0.40)	(0.40, 0.40, 0.50)	(0.90, 0.90, 1.00)	(0.45, 0.50, 0.65)
A4	(0.80, 0.95, 1.00)	(0.10, 0.15, 0.20)	(0.20, 0.20, 0.30)	(0.60, 0.70, 0.80)
A5	(0.35, 0.40, 0.50)	(0.70, 0.90,1.00)	(0.40, 0.40, 0.40)	(0.20, 0.30, 0.35)
A6	(0.50, 0.60, 0.70)	(0.80, 0.80, 0.90)	(0.40, 0.40, 0.60)	(0.10, 0.10, 0.20)
A7	(0.80, 0.80, 1.00)	(0.15, 0.20, 0.35)	(0.10, 0.10, 0.20)	(0.70, 0.70, 0.85)

6.6.2 Applying residual implications - $\mathcal{I} = \mathcal{I}_{I_{LK}}$

The triangle product operator $\triangleleft : (L_n(U))^2 \to L_n(U)$ is given as $\triangleleft \equiv \mathcal{F}_\wedge \circ \mathcal{I}$, considering $\mathcal{I} : (L_n(U))^2 \to L_n(U)$ operator obtained from Łukasiewicz n-DRI in Remark 5.2.1 and $\mathcal{F}_\wedge : L_n(U)^4 \to L_n(U)$ as the minimum operator in Eq.(19). Taking $k, j \in \{1, \ldots, n_2\}$ and $i \in \{1, \ldots, n_1\}$, the action of the $\triangleleft$-operator can be given by $n_1 \times n_2$-matrix whose elements $\mathbf{z}_{k,j} = \triangleleft(\mathbf{x}_{ki}, \mathbf{x}_{ji})$, are given as follows:

$$\mathbf{z}_{k,j} = \begin{cases} \left(\mathcal{F}_\wedge \circ \mathcal{I}\left(\mathbf{x}_{ki}, \mathbf{x}_{li}\right)_{(i=1\ldots4)}\right)_{(k,j=1\ldots7)}, \text{if } k \neq j \\ (1.0, 1.0, 1.0), \text{otherwise}. \end{cases} \tag{78}$$

And, the comparison of results is achieved whenever the ordered elements $\mathbf{z}_{(k,j)} \in L_4(U)$ are obtained.

See the action of Eq.(78) resulting on the matrix $L_{7\times7} = (\mathbf{z}_{(kj)})_{k,j=1\ldots7}$ presented in Table 4.

Table 4 – Action of the $\triangleleft$-operator in n-dimensional intervals.

	A1	A2	A3	A4	A5	A6	A7
A1	(1.00,1.00,1.00,1.00)	(0.40,0.80,0.90,0.95)	(0.40,0.65,1.00,1.00)	(0.40,0.50,1.00,1.00)	(0.55,0.60,0.90,1.00)	(0.70,0.75,0.8001.00)	(0.40,0.45,0.95,1.00)
A2	**(0.95,1.00,1.00,1.00)**	(1.00,1.00,1.00,1.00)	(0.35,0.70,1.00,1.00)	(0.40,0.95,1.00,1.00)	(0.50,1.00,1.00,1.00)	(0.65,0.95,1.00,1.00)	(0.50,0.90,0.95,1.00)
A3	(0.70,0.75,1.00,1.00)	(0.20,0.50,1.00,1.00)	(1.00,1.00,1.00,1.00)	(0.30,0.70,1.00,1.00)	(0.40,0.70,1.00,1.00)	(0.50,0.55,1.00,1.00)	(0.20,0.75,1.00,1.00)
A4	(0.55,0.90,1.00,1.00)	(0.35,0.90,0.90,1.00)	(0.35,0.80,1.00,1.00)	(1.00,1.00,1.00,1.00)	(0.45,0.55,1.00,1.00)	(0.40,0.65,1.00,1.00)	(0.85,0.90,1.00,1.00)
A5	(0.80,1.00,1.00,1.00)	(0.80,0.80,0.80,1.00)	(0.50,0.85,1.00,1.00)	(0.20,0.80,1.00,1.00)	(1.00,1.00,1.00,1.00)	(0.80,0.90,1.00,1.00)	(0.30,0.70,1.00,1.00)
A6	(0.90,1.00,1.00,1.00)	(0.60,0.80,0.95,1.00)	(0.60,0.70,1.00,1.00)	(0.30,0.70,1.00,1.00)	(0.80,0.80,0.90,1.00)	(1.00,1.00,1.00,1.00)	(0.35,0.60,1.00,1.00)
A7	(0.50,0.95,1.00,1.00)	(0.30,0.90,1.00,1.00)	(0.40,0.75,1.00,1.00)	(0.85,0.90,1.00,1.00)	(0.50,0.50,1.00,1.00)	(0.35,0.70,1.00,1.00)	(1.00,1.00,1.00,1.00)

Consider 1^{st} and 2^{nd} lines (A_1 and A_2 alternatives) in Table 3. So, the component $\mathbf{z}_{21}$ is given as:

$$\mathbf{z}_{21} = \mathcal{F}(\mathbf{y}_i)_{i=1\ldots4};$$
$$\mathbf{y}_i = \mathcal{I}_{I_{LK}}(\mathbf{x}_{2i}, \mathbf{x}_{1i}) = (\min(I(x_1, y_1), I(x_2, y_2), I(x_3, y_3)), \min(I(x_2, y_2), I(x_3, y_3)), I(x_3, y_3)) \in L_3(U).$$

It holds that:

$$\begin{aligned} \mathbf{y}_1 = \mathcal{I}_{I_{LK}}(\mathbf{x}_{21}, \mathbf{x}_{11}) &= \mathcal{I}_{I_{LK}}((0.85, 0.85, 0.9), (0.8, 0.85, 0.95)) \\ &= (\min(I_{LK}(0.85, 0.8), I_{LK}(0.85, 0.85), I_{LK}(0.9, 0.95)), \\ &\quad \min(I_{LK}(0.85, 0.85), I_{LK}(0.9, 0.95)), I_{LK}(0.9, 0.95)) \\ &= (\min(1, 1 - 0.85 + 0.8), \min(1, 1 - 0.85 + 0.85), \min(1, 1 - 0.9 + 0.95)), \\ &\quad \min(\min(1, 1 - 0.85 + 0.85)), \min(1, 1 - 0.9 + 0.95)), \min(1, 1 - 0.9 + 0.95)) \\ &= (0.95, 1.0, 1.0). \end{aligned}$$

$$\begin{aligned}
\mathbf{y}_2 &= \mathcal{I}_{I_{LK}}(\mathbf{x}_{22}, \mathbf{x}_{12}) = \mathcal{I}_{I_{LK}}((0.6, 0.7, 0.8), (0.7, 0.75, 0.8)) \\
&= (\min(I_{LK}(0.6, 0.7), I_{LK}(0.7, 0.75), I_{LK}(0.8, 0.8)), \\
&\min(I_{LK}(0.7, 0.75), I_{LK}(0.8, 0.8)), I_{LK}(0.8, 0.8)) \\
&= (\min(1, 1 - 0.6 + 0.7), \min(1, 1 - 0.7 + 0.75), \min(1, 1 - 0.8 + 0.8)), \\
&\min(1, 1 - 0.7 + 0.75), \min(1, 1 - 0.8 + 0.8)), \min(1, 1 - 0.8 + 0.8)) = (1.0, 1.0, 1.0).
\end{aligned}$$

$$\begin{aligned}
\mathbf{y}_3 &= \mathcal{I}_{I_{LK}}(\mathbf{x}_{23}, \mathbf{x}_{13}) = \mathcal{I}_{I_{LK}}((0.2, 0.2, 0.2), (0.65, 0.65, 0.8) \\
&= (\min(I_{LK}(0.2, 0.65), I_{LK}(0.2, 0.65), I_{LK}(0.2, 0.8)), \\
&\min(I_{LK}(0.2, 0.65), I_{LK}(0.2, 0.8)), I_{LK}(0.2, 0.8)) \\
&= (\min(1, 1 - 0.2 + 0.65), \min(1, 1 - 0.2 + 0.65), \min(1, 1 - 0.2 + 0.8)), \\
&\min(1, 1 - 0.2 + 0.65), \min(1, 1 - 0.2 + 0.8)), \min(1, 1 - 0.2 + 0.8)) = (1.0, 1.0, 1.0).
\end{aligned}$$

$$\begin{aligned}
\mathbf{y}_4 &= \mathcal{I}_{I_{LK}}(\mathbf{x}_{24}, \mathbf{x}_{14}) = \mathcal{I}_{I_{LK}}((0.15, 0.15, 0.15), (0.3, 0.3, 0.35)) \\
&= (\min(I_{LK}(0.15, 0.3), I_{LK}(0.15, 0.3), I_{LK}(0.15, 0.35)), \\
&\min(I_{LK}(0.15, 0.3), I_{LK}(0.15, 0.35)), I_{LK}(0.15, 0.35)) \\
&= (\min(1, 1 - 0.15 + 0.3), \min(1, 1 - 0.15 + 0.3), \min(1, 1 - 0.15 + 0.35)), \\
&\min(\min(1, 1 - 0.15 + 0.3), \min(1, 1 - 0.15 + 0.35)), \min(1, 1 - 0.15 + 0.35)) = (1.0, 1.0, 1.0).
\end{aligned}$$

Thus, we obtain $\mathbf{z}_{21} = \lhd(\mathbf{x}_{2i}, \mathbf{x}_{1i}) = \mathcal{F}_{\wedge} \circ \mathcal{I}_{I_{LK}}(\mathbf{x}_{2i}, \mathbf{x}_{1i})_{(i=1\ldots4)}$ as follows:

$$\mathbf{z}_{21} = (\wedge(0.95, 1.0, 1.0), \wedge(1.0, 1.0, 1.0), \wedge(1.0, 1.0, 1.0), \wedge(1.0, 1.0, 1.0)) = (0.95, 1.0, 1.0, 1.0).$$

And, the result component in Table 4 (shown in bold and placed at row-2; column-1) is a 4-dimensional interval $\mathbf{z}_{21}$ given as $\mathbf{z}_{(21)} = (0.95, 1.0, 1.0, 1.0) = \mathbf{z}_{21}$. The other components are obtained analogously.

6.6.2.1 Solving CIM-MCDM problem by preserving interval data

Since many elements in Table 4 of 4-dimensional intervals may not be comparable by the usual partial order $\leq_{L_4(U)}$, we consider the set $\mathcal{M}$ of aggregation-sequence given in Example 4.5.3, to apply the $\sqsubseteq_{[M]}$-order as the admissible linear order described in Proposition 4.5.2. Such interval data preserving strategy enables us to consider the uncertainty associated to input data modelling the possible indecision of specialist preferences. In addition, we remain capable to guarantee the comparison of all output interval data.

Thus, in this applications the use of aggregation operators (as the arithmetic mean) performed over the elements in Table 4 were not restricted, presenting the ordered values related to each component $\mathbf{z}_{(kl)} \in L_4(U)$ resulting from the action of $\triangleleft$-operator over data provided by evaluations in such MCDM problem.

See, for instance, the comparison between $\mathbf{z}_{(72)} = (\,0.3, 0.9, 1.0, 1.0\,)$ and $\mathbf{z}_{(27)} = (\,0.5, 0.9, 0.95, 1.0\,)$. Observe that they are incomparable w.r.t. the partial order $\leq_{L_n(U)}$, meaning that $\mathbf{z}_{(72)} \not\leq_{L_n(U)} \mathbf{z}_{(27)}$ and $\mathbf{z}_{(27)} \not\leq_{L_n(U)} \mathbf{z}_{(72)}$ hold. Then we apply the $\mathcal{M}$-aggregation sequence operator ($\mathcal{M} = \{M_1, M_2, M_3, M_4\}$) as presented in Example 4.5.3 in order to compare $\mathbf{z}_{(54)}$ and $\mathbf{z}_{(45)}$. In fact, $\mathbf{z}_{54} \sqsubset_{[M]} \mathbf{z}_{45}$ since $M_1\left(\mathbf{z}_{(54)}\right) = M_1\left(\mathbf{z}_{(45)}\right)$ but $M_2\left(\mathbf{z}_{(54)}\right) = 0.54 \leq 0.66 = M_2\left(\mathbf{z}_{(45)}\right)$. See, in bold, the calculation of the above results:

$$[M]\mathbf{z}_{(72)} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.50 & 0.15 & 0.15 & 0.20 \\ 0.20 & 0.20 & 0.30 & 0.30 \\ 0.10 & 0.40 & 0.10 & 0.40 \end{bmatrix} \begin{bmatrix} 0.20 \\ 0.80 \\ 1.00 \\ 1.00 \end{bmatrix} = \begin{bmatrix} \mathbf{0.75} \\ \mathbf{0.57} \\ 0.80 \\ 0.84 \end{bmatrix};$$

$$[M]\mathbf{z}_{(27)} = \begin{bmatrix} 0.25 & 0.25 & 0.25 & 0.25 \\ 0.50 & 0.15 & 0.15 & 0.20 \\ 0.20 & 0.20 & 0.30 & 0.30 \\ 0.10 & 0.40 & 0.10 & 0.40 \end{bmatrix} \begin{bmatrix} 0.50 \\ 0.90 \\ 0.95 \\ 1.00 \end{bmatrix} = \begin{bmatrix} \mathbf{0.75} \\ \mathbf{0.66} \\ 0.80 \\ 0.77 \end{bmatrix}.$$

Analogously, the other component comparisons are performed and we are able to conclude that $A7 > A2$.

Summarizing, in Table 5, all resulting comparisons are presented, where $\sqsubset_{[M]} \equiv \sqsubset$ and $\sqsupset_{[M]} \equiv \sqsupset$ are used as a reduced notation:

Table 5 – Applying admissible $\mathcal{M}$-order in the comparison degrees between alternatives in the Table 6.

$\mathbf{z}_{21} \sqsupset \mathbf{z}_{12}$	$\mathbf{z}_{12} \sqsubset \mathbf{z}_{21}$	$\mathbf{z}_{13} \sqsubset \mathbf{z}_{31}$	$\mathbf{z}_{14} \sqsubset \mathbf{z}_{41}$	$\mathbf{z}_{15} \sqsubset \mathbf{z}_{51}$	$\mathbf{z}_{16} \sqsubset \mathbf{z}_{61}$	$\mathbf{z}_{17} \sqsubset \mathbf{z}_{71}$
$\mathbf{z}_{31} \sqsupset \mathbf{z}_{13}$	$\mathbf{z}_{32} \sqsubset \mathbf{z}_{23}$	$\mathbf{z}_{23} \sqsupset \mathbf{z}_{32}$	$\mathbf{z}_{24} \sqsupset \mathbf{z}_{42}$	$\mathbf{z}_{25} \sqsupset \mathbf{z}_{52}$	$\mathbf{z}_{26} \sqsupset \mathbf{z}_{62}$	$\mathbf{z}_{27} \sqsupset \mathbf{z}_{72}$
$\mathbf{z}_{41} \sqsupset \mathbf{z}_{14}$	$\mathbf{z}_{42} \sqsubset \mathbf{z}_{24}$	$\mathbf{z}_{43} \sqsupset \mathbf{z}_{34}$	$\mathbf{z}_{34} \sqsubset \mathbf{z}_{43}$	$\mathbf{z}_{35} \sqsubset \mathbf{z}_{53}$	$\mathbf{z}_{36} \sqsubset \mathbf{z}_{63}$	$\mathbf{z}_{37} \sqsubset \mathbf{z}_{73}$
$\mathbf{z}_{51} \sqsupset \mathbf{z}_{15}$	$\mathbf{z}_{52} \sqsubset \mathbf{z}_{25}$	$\mathbf{z}_{53} \sqsupset \mathbf{z}_{35}$	$\mathbf{z}_{54} \sqsubset \mathbf{z}_{45}$	$\mathbf{z}_{45} \sqsupset \mathbf{z}_{54}$	$\mathbf{z}_{46} \sqsupset \mathbf{z}_{64}$	$\mathbf{z}_{47} = \mathbf{z}_{74}$
$\mathbf{z}_{61} \sqsupset \mathbf{z}_{16}$	$\mathbf{z}_{62} \sqsubset \mathbf{z}_{26}$	$\mathbf{z}_{63} \sqsupset \mathbf{z}_{36}$	$\mathbf{z}_{64} \sqsubset \mathbf{z}_{46}$	$\mathbf{z}_{65} \sqsubset \mathbf{z}_{56}$	$\mathbf{z}_{56} \sqsupset \mathbf{z}_{65}$	$\mathbf{z}_{57} \sqsubset \mathbf{z}_{75}$
$\mathbf{z}_{71} \sqsupset \mathbf{z}_{17}$	$\mathbf{z}_{72} \sqsubset \mathbf{z}_{27}$	$\mathbf{z}_{73} \sqsupset \mathbf{z}_{37}$	$\mathbf{z}_{74} = \mathbf{z}_{47}$	$\mathbf{z}_{75} \sqsupset \mathbf{z}_{57}$	$\mathbf{z}_{76} \sqsupset \mathbf{z}_{67}$	$\mathbf{z}_{67} \sqsubset \mathbf{z}_{76}$

Moreover, one can easily observe that $\mathbf{z}_{i1} \sqsupset \mathbf{z}_{1i}$, for all $i \in \{1, \ldots, 7\}$. Then A_1 is the superior CIM software alternative by comparing it with other alternatives. The same analysis can be performed to other alternatives and the following conclusion is finally reached:

$$A1 > A3 > A6 > A5 > A4 = A7 > A2.$$

6.6.2.2 Solving CIM-MCDM problem by applying aggregation operators

Regarding the previous data from Table 4, the arithmetic means given as

$$\mathbf{z}_{k,j} = \left(\frac{1}{4} \sum_{k=1}^{4} \mathcal{F}_{\wedge} \circ \mathcal{I}_{I_{LK}} \left(\mathbf{x}_{ki}, \mathbf{x}_{li} \right) \right)_{k,j=1\dots7}, \tag{79}$$

is performed, resulting in the data shown in Table 6. Thus, a new comparison is obtained.

Table 6 – n-dimensional implication degree calculation.

	A1	**A2**	**A3**	**A4**	**A5**	**A6**	**A7**
A1	1.0000	0.7625	0.7625	0.7250	0.7625	0.8125	0.7000
A2	0.9875	1.0000	0.7625	0.8375	0.8750	0.9000	0.8375
A3	0.8625	0.6750	1.0000	0.7500	0.7750	0.7625	0.7375
A4	0.8625	0.7875	0.7875	1.0000	0.7500	0.7625	0.9375
A5	0.9500	0.8500	0.8375	0.7500	1.0000	0.9250	0.7500
A6	0.9750	0.8375	0.8250	0.7500	0.8750	1.0000	0.7375
A7	0.8625	0.8000	0.7875	0.9375	0.7500	0.7625	1.0000

Based on data analysis in Table 6, the conclusion reached is:

$$A1 > A3 > A6 > A4 = A5 = A7 > A2.$$

Despite the simplification provided by the arithmetic aggregation operator in the second strategy, the corresponding comparison between alternatives $A4$ and $A5$ cannot be explicit ($A4 = A5$). However, the comparison $A5 > A4$ is achieved in the first strategy, which seems more suitable for these two distinct 4-dimensional intervals.

To sum up, the use of n-dimensional intervals in $L_n(U)$ in modelling MCDM problems seems more intuitively, since their inherent ordered-components are relevant to perform comparisons. Moreover, the use of admissible linear order in $L_n(U)$ can provide more detailed comparison analysis when we are dealing with MCDM problems.

7 EXPLORING n-DIMENSIONAL FUZZY IMPLICATIONS IN APPROXIMATE REASONING

This chapter, first results in the extension of the basic concepts of approximate reasoning (AR) are considered, by using n-dimensional intervals.

The idea of linguistic fuzzy models imitating the human way of thinking was proposed by Zadeh in his pioneering work (ZADEH, 1973). The term AR refers to methods and methodologies which enable reasoning with imprecise inputs to obtain meaningful outputs (DRIANKOV; HELLENDOORN; REINFRANK, 2013).

By employing fuzzy logic inference schemes as generalized modus ponens (GMP) or generalized modus tollens (GMT), one can carry out AR when the knowledge base consists of fuzzy IF-THEN rules (LIU, 2010).

Approximate reasoning is one of the best known application areas of fuzzy logic, wherein from imprecise inputs and fuzzy premises or rules we often obtain conclusions. Encompassing a wide variety of inference schemes, we consider the n-dimensional extensional of basic concepts which generalize fuzzy conditional rules, based on n-dimensional fuzzy implications, in order to solve a DMP (TURKSEN, 1989).

According with (BACZYŃSKI; JAYARAM, 2008), inference in AR is in sharp contrast to inference in classical logic - in the former the consequence of a given set of fuzzy propositions depends on an essential way of the meaning attached to these fuzzy propositions. Thus, inference in AR is a computation with fuzzy sets representing the meaning of a certain set of fuzzy propositions.

An inference scheme proposed under AR is validated or assessed mainly based on the reasonableness of inference (effectiveness) and the complexity of the algorithm (efficiency). For example, in the area of DM, AR techniques are employed for their inferential capabilities following the basic rules of GMP.

Fuzzy implications play an important role in all the different realizations of inference schemes since the knowledge base consists of fuzzy conditionals.

Boolean implications are usually employed in inference schemas such as, GMP and GMT, where the reasoning is done with statements or propositions whose truth-values are two-valued. Analogously, fuzzy implications play a similar role in the generalizations of the above inference schemas, where reasoning is done with fuzzy statements whose truth-values lie in $[0, 1]$ instead of $\{0, 1\}$ (BACZYŃSKI; JAYARAM, 2008).

Fuzzy implication is employed to relate fuzzy propositional formulae in fuzzy logic inference schemes. For example, if A, B are any fuzzy logic propositional formulae, then $A \to B$ is called a *fuzzy conditional statement* or more commonly as a fuzzy IF-THEN rule and it is again interpreted as "A implies B".

This construction can be extended to n-dimensional fuzzy AR, considering both aspects:

(i) n-dimensional intervals and fuzzy statements

An expression of the form "$\mathbf{x}$ is A"is termed as a fuzzy statement, where A is a n-dimensional fuzzy set on the n-dimensional simplex $L_n(U)$, with reference to the context. Thus, we can say that the above statement can be interpreted as follows:

- Let "$\mathbf{x}$ is A"and also that $\mathbf{x}$ assumes the precise value, let us say, $\mu_A(\mathbf{x}) = u \in L_n(U)$, the domain of A. Then the truth value of the above fuzzy statement is obtained as $t(\mathbf{x} \text{ is } A) = A(u)$. Thus, the greater the membership degree of $\mathbf{x}$ in the concept A is, the higher the truth value of the fuzzy statement.

While in the above case a fuzzy statement was looked upon as a fuzzy proposition to be evaluated based on some precise information, it can also be used to express something precise when the only information regarding the variable $\mathbf{x}$ is imprecise.

(ii) n-dimensional intervals and IF-THEN rules

We can also interpret an n-dimensional fuzzy statement as a linguistic statement on the suitable domain $L_n(U)$. Then A represents a concept and hence can be thought of as a linguistic value. Then a symbol $\mathbf{x}$ can assume or be assigned to a linguistic value. Then a linguistic statement "$\mathbf{x}$ is A"is interpreted as the linguistic variable $\mathbf{x}$ taking the linguistic value A.

7.1 n-Dimensional intervals and inference schemes in approximate reasoning

Analogous to the fuzzy logic, inference schemes in approximate reasoning on the n-dimensional simplex domain can consider the following structures in the fuzzy rules of deduction.

The fuzzy rules of deduction generalize the corresponding classical two-valued logic rules of deduction, which are frequently related to:

- GMP, considering an inequality explicitly by a conjunction, which is defined as a t-norm, and a fuzzy implication;
- GMT, deriving into an inequality involving three logical operators: a conjunction taking as a t-norm, a fuzzy implication and a negation, as the strong standard negation; and
- Hypothetical syllogism, as a composition of fuzzy implications, frequently verifying properties from the class of R-implications.

7.1.1 Generalized modus ponens

This thesis studies inference schemes realizing GMP in AR related to n-dimensional fuzzy approaches, considering:

1. combination-projection principle, of which the compositional rule of inference (CRI) of (ZADEH, 1973);
2. similarity between inputs and antecedents and the subsequent modification of the consequent, usually known as similarity based reasoning (SBR) (DUBOIS, 1985).

In order to structure fuzzy rules in inference schemes based on the inference pattern called GMP, we consider the following:

- the fuzzy rule of the form "IF $\mathbf{x}$ is A THEN $\mathbf{y}$ is B", and
- the fact "$\mathbf{x}$ is A";
- Thus, GMP allows the following conclusion to be drawn "$\mathbf{y}$ is B" when $A, A' \in \mathcal{F}$ and $B, B' \in \mathcal{F}$.

However, A' is not necessarily identical to A and so, B' is also not necessarily identical to B.

Next section extends the AR results presented in (BACZYŃSKI; JAYARAM, 2008) in the n-dimensional fuzzy logic context.

7.2 Compositional rule of inference on $L_n(U)$

This section describes the application of compositional rules of inference (CRI) systems on $L_n(U)$.

Let $A_{L_n(U)} \in FS_n(\chi)$ be n-dimensional fuzzy set related to a universe χ and given by the membership function $\mu_{A_{L_n(U)}} : \chi \to L_n(U)$. Consider $\mu_{A_{L_n(U)}} \equiv A$ in order to reduce denotation in the following results.

Let $A_1, \ldots, A_k \in FS_n(\chi)$, related to the universe $\chi = \{\mathrm{x}_i : i \in \mathbb{N}_l\}$, meaning that $\#\chi = l$. The cartesian product of these n-DFS is given in the next definition.

Definition 7.2.1 *Let $\chi = \{\mathrm{x}_i \colon i \in \mathbb{N}_l\}$ be a universe set. Consider $A_1, \ldots, A_m \in FS_n(\chi)$ such that for each $j \in \mathbb{N}_m$, clearly $\mathcal{A}$ is an nDFS when $A_1 \leq \ldots \leq A_m$ holds. Then, the cartesian product of the n-dimensional fuzzy $A_1, \ldots, A_m$ is denoted as $\mathcal{A} = A_1 \times \ldots \times A_m$ and defined by*

$$\mathcal{A}(\mathrm{x}) = (A_1, \ldots, A_m)(\mathrm{x}) = (A_1(\mathrm{x}), \ldots, A_m(\mathrm{x})), \forall \mathrm{x} \in \chi.$$

Notice that the $\mathcal{A}(\mathrm{x})$ can be seen as a matrice on $L_n(U)$ of $l \times m$ dimension where $\mathcal{A}_{ij} = \mathcal{A}_j(\chi_i)$, $\mathcal{A}_{ij} \in L_n(U)$.

Definition 7.2.2 *Let $\mathcal{A}$ a matrice on $L_n(U)$ in the conditions of Definition 7.2.1 and a function $\mathcal{P} : (L_n(U))^k \to L_n(U)$ define the n-DFS on χ, $\mathcal{P}\mathcal{A} : \chi \to L_n(U)$.*

$$\mathcal{P}\mathcal{A}(\mathrm{x}) = \mathcal{P}(A_1, \ldots, A_m)(\mathrm{x}) = \mathcal{P}(A_1(\mathrm{x}), \ldots, A_m(\mathrm{x})), \forall \mathrm{x} \in \chi.$$

Example 7.2.1 *Let $\chi = \{1, 2\}$, meaning that $i \in \mathbb{N}_2$. For each $j \in \mathbb{N}_4$, consider the 4-dimensional fuzzy sets A_1, A_2, A_3 defined as follows*

$$\pi_j(A_1(\mathrm{x}_i)) = \frac{\mathrm{x}_i}{n-j+2} \Rightarrow A_1 = \{(0.2, 0.25, 0.\overline{3}, 0.5), (0.4, 0.5, 0.\overline{6}, 1.0)\}; \tag{80}$$
$$\pi_j(A_2(\mathrm{x}_i)) = \frac{\mathrm{x}_i}{n-j+3} \Rightarrow A_2 = \{(0.1\overline{6}, 0.2, 0.25, 0.5), (0.\overline{3}, 0.4, 0.5, 0.\overline{6})\}; \tag{81}$$
$$\pi_j(A_3(\mathrm{x}_i)) = \frac{\mathrm{x}_i}{n-j+4} \Rightarrow A_3 = \{(0.14, 0.1\overline{6}, 0.2, 0.25), (0.28, 0.\overline{3}, 0.4, 0.5)\}. \tag{82}$$

Now, taking the associative operators $\bigwedge, \bigvee : (L_n(U))^3 \to L_2(U)$ we have that

$$\begin{aligned}
\bigwedge(A_1, A_2, A_3)(\mathrm{x}_i) &= \bigwedge(A_1(\mathrm{x}_1), A_2(\mathrm{x}_1), A_3(\mathrm{x}_1)) \\
&= \bigwedge(A_1, A_2, A_3) = ((0.14, 0.1\overline{6}, 0.2, 0.25), (0.28, 0.\overline{3}, 0.4, 0.5)); \\
\bigvee(A_1, A_2, A_3)(\mathrm{x}_i) &= \bigvee(A_1(\mathrm{x}_2), A_2(\mathrm{x}_2), A_3(\mathrm{x}_2)) \\
&= \bigvee(A_1, A_2, A_3) = ((0.2, 0.25, 0.\overline{3}, 0.5), (0.4, 0.5, 0.\overline{6}, 1.0)).
\end{aligned}$$

Definition 7.2.3 *(ZANOTELLI et al., 2019) Let χ_1, χ_2 be finite, nonempty sets and $\mathcal{A} = \{A_1, A_2\}$. The cartesian product of the n-DFS A_1 and A_2 with respect to an n-dimensional t-norm $\mathcal{T}$ is an n-dimensional fuzzy set on $FS_n(\chi_1 \times \chi_2)$ defined as follows:*

$$\mathcal{T}(A_1, A_2)(\mathrm{x}_1, \mathrm{x}_2) = \mathcal{T}(A_1(\mathrm{x}_1), A_2(\mathrm{x}_2)), \quad \forall \mathrm{x}_1 \in \chi_1, \mathrm{x}_2 \in \chi_2.$$

Based on Definitions 7.2.1 and 7.2.3, an IF-THEN rule is represented by a n-dimensional fuzzy relation $\mathcal{R}_{\mathcal{I}}(A_1, A_2) : (L_n(U))^2 \to L_n(U)$ given as:

$$\mathcal{R}_{\mathcal{I}}(A_1, A_2)(\mathrm{x}_1, \mathrm{x}_2) = \mathcal{I}(A_1(\mathrm{x}_1), A_2(\mathrm{x}_2)), \quad \forall \mathrm{x}_1 \in \chi_1, \mathrm{x}_2 \in \chi_2, \tag{83}$$

where $\mathcal{I}$ is usually an n-dimensional fuzzy implication and A_1, A_2 are fuzzy sets on their respective universe domains χ_1, χ_2. Therefore, given a fact "x_1 is A_1", the inferred output "x_2 is A_2'" is obtained as sup-$\mathcal{T}$ composition of $A_1'(\mathrm{x}_1)$ and $R(\mathrm{x}_1, \mathrm{x}_2)$, as follows:

$$A_2'(\mathrm{x}_2) = (A_1' \circ^{\mathcal{T}} R)(\mathrm{x}_2) \quad = \bigvee_{\mathrm{x}_1 \in \chi_1} \mathcal{T}(A_1'(\mathrm{x}_1), R(\mathrm{x}_1, \mathrm{x}_2)) = \bigvee_{\mathrm{x}_1 \in \chi_1} \mathcal{T}(A_1'(\mathrm{x}_1), \mathcal{I}(A_1(\mathrm{x}_1), A_2(\mathrm{x}_2))).$$

Let A_1, A_2, A_3 be n-DFS on their respective universe domains χ_1, χ_2, χ_3. So, considering the two following cases.

1. First, considering a Single-Input Single-Output (SISO) system, when the input A_1' is a singleton n-dimensional fuzzy set given by Eq.(14) attaining normality at an $x_1' \in \chi_1$, then the related output constructing is obtained as follows:

$$\begin{aligned} A_2'(\mathrm{x}_2) &= A_1'(\mathrm{x}_1) \circ^{\mathcal{T}} R(\mathrm{x}_1, \mathrm{x}_2) \\ &= \bigvee_{\mathrm{x} \in \chi} \mathcal{T}(A_1'(\mathrm{x}_1), R(\mathrm{x}_1, \mathrm{x}_2)) \\ &= \mathcal{T}(A_1'(\mathrm{x}'_1), R(\mathrm{x}'_1, \mathrm{x}_2)) \\ &= \mathcal{T}(/1/, R(\mathrm{x}'_1, \mathrm{x}_2)) = R(\mathrm{x}'_1, \mathrm{x}_2). \end{aligned}$$

2. And, in the another case, considering the rule-base in a Multi-Input Single-Output (MISO) system, the relation R is given by

$$R(\mathrm{x}_1, \mathrm{x}_2, \mathrm{x}_3) = \mathcal{I}(A_1(\mathrm{x}_1) \odot A_2(\mathrm{x}_2), A_3(\mathrm{x}_3)), \tag{84}$$

where $\odot$, called the n-dimensional antecedent combiner, it is usually an n-dimensional t-norm is applied. Thus, we have

$$A_1(\mathrm{x}_1) \odot A_2(\mathrm{x}_2) = \mathcal{T}(A_1, A_2)(\mathrm{x}_1, \mathrm{x}_2),$$

meaning that, it is the cartesian product of the n-DFS A_1, A_2 with respect to the n-dimensional t-norm $\mathcal{T}$. So, given a multiple-input (A_1', A_2') the inferred output A_3', taking the $\sup$-$\mathcal{T}$ composition, is given by the following expression

$$A_3' = (A_1', A_2') \circ^{\mathcal{T}} ((A_1, A_2) \to A_3). \tag{85}$$

Then, by applying results of Eq.(85), for all $\mathbf{x} \in \chi_3$, we obtain the following expression for an output in the IF-THEN base-rule in a MISO system:

$$A_3'(\mathbf{x}) = \bigvee_{(\mathrm{x}_1, \mathrm{x}_2) \in \chi_1 \times \chi_2} \mathcal{T}(A_1'(\mathrm{x}_1) \odot A_2'(\mathrm{x}_2), \mathcal{I}(A_1(\mathrm{x}_1) \odot A_2(\mathrm{x}_2), A_3(\mathrm{x}_3))), \forall \mathrm{x}_{\in} \chi_3. \tag{86}$$

In the following, an example exploring the structure presented in Eq.(86).

7.2.1 Exemplification of IF-THEN base-rule in MISO n-dimensional fuzzy system

Let A, B and C be n-DFS given as follows:

$$A = \{(0.85, 0.90, 0.95), (0.75, 0.80, 0.85), (0.65, 0.70, 0.75), (0.65, 0.70, 0.75)\},$$
$$B = \{(0.95, 1, 1), (0.55, 0.60, 0.65), (0.75, 0.80, 0.85)\},$$
$$C = \{(0.05, 0.1, 0.15), (0.05, 0.1, 0.15), (0.15, 0.2, 0.25)\}.$$

Or $\chi_A = \{a_1, \ldots, a_4\}, \chi_B = \{b_1, b_2, b_3\}$ and $\chi_C = \{c_1, c_2, c_3\}$. The cartesian product between n-dimensional fuzzy sets A and B with respect to $\mathcal{T}_P(\mathbf{x}, \mathbf{y}) = \mathbf{xy}$, is given as follows:

$$\mathcal{T}_P(A, B) = \begin{pmatrix} (T_P(A(a_1), B(b1)) & T_P(A(a_1), B(b2)) & T_P(A(a_1), B(b3)) \\ (T_P(A(a_2), B(b1)) & T_P(A(a_2), B(b2)) & T_P(A(a_2), B(b3)) \\ ((T_P(A(a_3), B(b1)) & T_P(A(a_3), B(b2)) & T_P(A(a_3), B(b3)) \\ (T_P(A(a_4), B(b1)) & T_P(A(a_4), B(b2)) & T_P(A(a_4), B(b3)) \end{pmatrix}$$

$$\mathcal{T}_P(A, B) = \begin{pmatrix} (0.8075, 0.9000, 0.9500) & (0.4675, 0.5400, 0.6175) & (0.6375, 0.7200, 0.8075) \\ (0.7125, 0.8000, 0.8500) & (0.4125, 0.4800, 0.5525) & (0.5625, 0.6400, 0.7225) \\ (0.6175, 0.7000, 0.7500) & (0.3575, 0.4200, 0.4875) & (0.4875, 0.5600, 0.6375) \\ (0.6175, 0.7000, 0.7500) & (0.3575, 0.4200, 0.4875) & (0.4875, 0.5600, 0.6375) \end{pmatrix}$$

In addition, let $I_{KD}, I_{RC}, I_{LK} : U^2 \to U$ be implications given by the expressions:

- I_{KD} be the Kleene-Dienes implication $I_{KD}(x, y) = \max(1 - x, y)$,
- I_{RC} be the Reichenbach implication $I_{RC}(x, y) = 1 - x + xy$,
- I_{LK} be the Lukasiewicz implication $I_{LK}(x, y) = \min(1, 1 - x + y)$.

We will combined the n-DI $\mathcal{I}_3 : L_n(U)^2 \to L_n(U)$ is given as $\mathcal{I}_3(\mathbf{x}, \mathbf{y}) = \widetilde{I_{KD}, I_{RC}}, I_{LK}(\mathbf{x}, \mathbf{y})$. So taking a the relation R defined in Eq.(84) own

$$R(z_1){=}R(z_2){=}\begin{pmatrix} (0.0500, 0.1000, 0.1925) & (0.4134, 0.5140, 0.6026) & (0.2425, 0.3800, 0.5125) \\ (0.1500, 0.2000, 0.2875) & (0.4751, 0.5680, 0.6494) & (0.3275, 0.4600, 0.5875) \\ (0.2500, 0.3000, 0.3825) & (0.5369, 0.6220, 0.6961) & (0.4125, 0.5400, 0.6625) \\ (0.2500, 0.3000, 0.3825) & (0.5369, 0.6220, 0.6961) & (0.4125, 0.5400, 0.6625) \end{pmatrix}$$

$$R(z_3){=}\begin{pmatrix} (0.1500, 0.2000, 0.2500) & (0.4751, 0.5680, 0.6494) & (0.3425, 0.4800, 0.6125) \\ (0.1500, 0.2000, 0.2875) & (0.5304, 0.6160, 0.6906) & (0.4275, 0.5600, 0.6875) \\ (0.2500, 0.3000, 0.3825) & (0.5856, 0.6640, 0.7319) & (0.5125, 0.6400, 0.7625) \\ (0.2500, 0.3000, 0.3825) & (0.5856, 0.6640, 0.7319) & (0.5125, 0.6400, 0.7625) \end{pmatrix}$$

Let $A' = [(0,0,0),(0,0,0),(1,1,1),(0,0,0)]$, $B' = [(0,0,0),(1,1,1),(0,0,0)]$ be the given n-DFS singleton inputs. Then $K = A' \odot B'$ meaning that

$$K = \mathcal{T}_P(A', B') = \begin{pmatrix} (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \\ (0,0,0) & (1,1,1) & (0,0,0) \\ (0,0,0) & (0,0,0) & (0,0,0) \end{pmatrix}$$

By $\sup - \mathcal{T}_M$ composition, we have $C' = \mathcal{T}_P(A', B') \circ^{\mathcal{T}_M} \widetilde{I_{KD}, I_{RC}}, I_{LK}(\mathcal{T}_P(A,B), C)$.

Then, the result is expressed in the following output:

$$\begin{aligned} C' &= (T_P(R(z_1), K), T_P(R(z_2), K), T_P(R(z_3), K)) \\ &= ((0.5369, 0.6220, 0.6961), (0.5369, 0.6220, 0.6961), (0.5856, 0.6640, 0.7319)) \end{aligned}$$

8 CONCLUSION

In the following, conclusions are presented, summarizing main results achieved in the development of the research, and also describing further work.

8.1 Relevance and originality of the thesis

The study of n-dimensional intervals provide a new possibility to model not only the imprecision in the calculations, but also to describe the (possible repeated) uncertainty opinion provided by many specialists related to their preference relations faced to multi-criteria data. In these contexts, n-DFL approach contributes for the development of applications based on multi-expert decision making (MEDM) problems.

Such ordered structures are extensions of FS as old as HFS but less explored, in theoretical and applied research. Thus, the new investigations promoted by this thesis research can contribute to consolidate this extension of FL in both senses:

- theoretical foundations of main results in n-DFL,
- main results considering the extended study of n-dimensional fuzzy implications, their main properties and classes contributing to explore other research fields taking advantages of logical structure of n-DFL:
  - formalizing IF-THEN rules of approximate reasoning;
  - structuring new methods to measure similarity and consensus n-DFS;
  - extracting parameters as correlation coefficient and entropy supporting the data analysis of applications;
  - analysing the robustness related to fuzzy connectives in n-DFL.

To sum up, the methodology considered to achieve the main results of this thesis can be extended to the above activities.

8.2 Main results

Main results are highlighted below, considering the presented theoretical research based on n-dimensional fuzzy sets and in addition, the main related works.

8.2.1 Consolidating the research of fuzzy implications on $L_n(U)$

The extension methods of fuzzy implications from U to $L_n(U)$ are investigated in this research work, applying dual and conjugate constructors.

- Representable n-dimensional fuzzy implications on $L_n(U)$:

  This thesis studies the representability, as a generation of n-dimensional fuzzy implications which can be expressed by fuzzy implications. The concepts underlying the expression of representable n-dimensional t-norms and t-conorms in conjunction with representable (strong) n-dimensional fuzzy negations and their interrelationship with the n-dimensional aggregation operator are discussed.

- Main properties of fuzzy implication on $L_n(U)$:

  The conditions under which main properties are preserved based on their representability from U^n to $L_n(U)$ are also studied. Some results in the class of (S, N)-implications, QL-implications and R-implications obtained from t-representable conorms and norms, including involutive n-dimensional fuzzy negations on $L_n(U)$ are also studied.

- Implications and duality property on $L_n(U)$:

  n-dimensional fuzzy implications are studied considering duality constructions, as n-dimensional fuzzy coimplications and their main examples.

- Automorphisms and conjugate operators on $L_n(U)$:

  Based on the action of (representable) automorphisms on $L_n(U)$ n-dimensional fuzzy implications can be obtained as fuzzy conjugation operators preserving their main properties.

8.2.2 Strengthening results of fuzzy implication classes on $L_n(U)$

- Consolidating the Study of (S, N)-Implications on $L_n(U)$

  n-dimensional fuzzy implications are studied considering duality and conjugation operators. Additionally, the conditions over which main properties are preserved based on their representability from U^n to $L_n(U)$ are also presented. Some results in the class of (S, N)-implications obtained from representable t-conorms and involutive n-dimensional fuzzy negations on $L_n(U)$ are discussed.

- Consolidating the study of QL-implications on $L_n(U)$

  As another contribution, this work introduces the definition of n-dimensional fuzzy QL-implications. Focusing on the QL-implication class, representable n-dimensional t-conorms in conjunction with representable n-dimensional fuzzy negations are considered, showing that intrinsic properties of QL-implications are preserved from U^n to $L_n(U)$. By considering projection functions and degenerate elements, our results make use of the representability of such connectives on $L_n(U)$. And, by considering the action of n-dimensional automorphisms and strong fuzzy negation on $L_n(U)$, the conjugate and dual operators of n-dimensional QL-implications are studied.

- Extending the study of R-implication on $L_n(U)$

  R-implications are studied on $L_n(U)$ considering the conditions under which main properties are preserved, and their representability from U to $L_n(U)$ are also investigated. Some results in the class of n-dimensional R-implications obtained from t-representable norms on $L_n(U)$ and the residuation property are discussed.

  As a main contribution, properties characterizing the class of R-implications on $L_n(U)$ are studied. We also provide a characterization of n-RDI based on left-continuous n-DT. The methodology obtaining n-dimensional t-norm from n-dimensional implications using the residuation principle is presented. Such residuation property, given by a left-continuous n-DT, provides an exhaustive study of n-dimensional R-implications based on the discussion of the exchange principle and the ordering properties.

  An illustration on solving a CIM-MCDM application is extended from hesitant fuzzy sets to n-dimensional fuzzy sets.

- Introduction of the basic concepts of the AR study related to fuzzy implications on $L_n(U)$

  This thesis introduced the study of aggregators and implications for working with AR. In such context, introductory concepts of statements and rules extending AR were extended to the n-dimensional fuzzy interval approach, considering n-dimensional fuzzy aggregators. In sequence, an example using n-dimensional fuzzy implication in conjunction with AR. Ongoing work considers extending AR by using other classes of n-dimensional fuzzy implications.

8.2.3 Reporting the main results

In the following, the main themes related to the topic of this doctoral thesis are listed and other publications related to collaborators in the research groups of

FFMMFCC/UFPE, LUPS/UFPEL and LoLITA /UFRN achieved during the doctoral period are presented.

8.2.3.1 Contributions reported in journals publications

1. Rosana Zanotelli, Renata Reiser, Adenauer Yamin and Benjamín Bedregal, Intuitionistic Fuzzy Differences: Robustness and Duality Analysis, *Multiple-Valued Logic and Soft Computing* (ZANOTELLI et al., 2018).

2. Lidiane Costa, Mônica Matzenauer, Rosana Zanotelli, Mateus Nascimento, Alice Finger, Renata Reiser, Adenauer Yamin and Maurıcio Pilla, Analysing Fuzzy Entropy via Generalized Atanassov's Intuitionistic Fuzzy Indexes *Mathware & Soft Computing*, (COSTA et al., 2017).

3. Renata Reiser, Rosana Zanotelli, Simone Costa, Luciana Foss and Benjamín Bedregal, Robustness of f- and g-generated Fuzzy (Co)Implications: The Yager's (Co)Implication Case Study, *Electronic Notes in Theoretical Computer Science* (REISER et al., 2016).

8.2.3.2 Contributions reported in events and congress proceedings

1. Rosana Zanotelli, Renata Reiser and Benjamín Bedregal, Study on n-Dimensional R-implications, *Conference of the International Fuzzy Systems Association and the European Society for Fuzzy Logic and Technology (EUSFLAT 2019)* (ZANOTELLI; REISER; BEDREGAL, 2019/08). Considering the conditions under which main properties are preserved, and their representability from U to $L_n(U)$ is also presented. Some results in the class of n-dimensional R-implications obtained from t-representable norms on $L_n(U)$ are discussed.

2. Rosana Zanotelli, Renata Reiser and Benjamín Bedregal, Towards Inference Schemes in Approximate Reasoning using n-Dimensional Fuzzy Logic, *Proceedings of WEIT 2019 - Workshop-Escola de Informática Teórica* (ZANOTELLI et al., 2019). Introducing an extension of the approximate reasoning area, which refers to methods and methodologies that allow you to reason with inaccurate inputs to obtain meaningful results. With that in mind, in this paper the approximation reasoning was extended to the n-dimensional approach where we introduce an example using n-dimensional fuzzy aggregators and n-dimensional fuzzy implications in conjunction with AR.

3. Rosana Zanotelli, Renata Reiser and Benjamín Bedregal, n-Dimensional Intervals and Fuzzy S-implications, *2018 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)* (ZANOTELLI; REISER; BEDREGAL, 2018a). Considering a relevant extension of fuzzy sets, n-dimensional fuzzy sets consider mem-

bership degrees as ordered n-tuples on $U = [0,1]$, providing n-dimensional intervals on $L_n(U)$. As our main contribution, n-dimensional fuzzy implications are studied considering duality and conjugation operators. Additionally, the conditions over that main properties are preserved based on their representability from U^n to $L_n(U)$ are also presented. Some results in the class of S-implications obtained from t-representable conorms and involutive n-dimensional fuzzy negations on $L_n(U)$ are discussed.

4. Rosana Zanotelli, Renata Reiser and Benjamín Bedregal, Towards the study of main properties of n-Dimensional QL-implicators, *Proceedings of CBSF 2018 - Congresso Brasileiro de Sistemas Fuzzy* (ZANOTELLI; REISER; BEDREGAL, 2018b). Considered as a relevant extension of fuzzy sets, n-dimensional fuzzy sets consider membership degrees as ordered n-tuples on U, providing n-dimensional intervals on $L_n(U)$. As main contribution, n-dimensional QL-implicators are studied considering duality and conjugation operators. Some results in the class of QL-implicators obtained from t-representable t-norms and t-conorms and involutive n-dimensional fuzzy negations on $L_n(U)$ are discussed. Additionally, the conditions over that main properties are preserved based on their representability from U^n to $L_n(U)$ are also presented.

5. Renata Reiser, Rosana Zanotelli, Lidiane Costa, Monica Matzneuer, Benjamín Bedregal and Ivan Mezzomo, A class of fuzzy implications obtained from triples of fuzzy implications, *2017 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)* (REISER et al., 2017). Considers the to study the class $\mathcal{I}_{I,I_1,I_2}(U)$ of fuzzy implication obtained by a triple (I, I_1, I_2) of fuzzy implications. Thus, this paper discusses under which conditions such functions preserve the main properties of fuzzy implications. In addition, by conjugate fuzzy implications it is shown that an $\mathcal{I}_{I,I_1,I_2}(U)$ fuzzy implication can be preserved by action of an order automorphism. Finally, we introduce the family $\mathcal{I}^{(k)}_{I,I_1,I_2}(U)$ of fuzzy implications obtained by taking the extended classes of $\mathcal{I}^{(k)}$-implications verifying both generalized properties, exchange principle and distributivity in addition to, their dual construction is also considered.

6. Rosana Zanotelli, Renata Reiser, Benjamín Bedregal and Ivan Mezzomo, Main Properties of n-Dimensional Fuzzy Implications, *Proceedings of WEIT 2017 - Workshop-Escola de Informática Teórica* (ZANOTELLI et al., 2017). Investigates a special extension from U^n – the n-dimensional fuzzy implication on $L_n(U)$, summarizing the class of such functions which are extended continuous functions. The main properties of fuzzy implication on U^n are preserved by n-representable fuzzy implications on $L_n(U)$, mainly related to the analysis of degenerate ele-

ments and projections. It was observed that the main properties as representability and continuity on $L_n(U)$ are preserved.

7. Rosana Zanotelli, Lidiane da Silva, Miriam Born, Renata Reiser and Adenauer Yamin, Generalized Atanassov's Intuitionistic Fuzzy Index and Conjugate with S-implications, *CNMAC 2016 - Congresso Nacional de Matemática Aplicada e Computacional, Proceeding Series of the Brazilian Society of Computational and Applied Mathematics* (ZANOTELLI et al., 2017). Extends the study of properties related to the Generalized Atanassov's Intuitionistic Fuzzy Index, by considering the concept of conjugate fuzzy implications, mainly interested in the class of S-implications.

8. Rosana Zanotelli, Wilson Cardoso, Renata Reiser and Luciana Foss, Robustness on the class of fuzzy difference operators, *CNMAC 2016 - Congresso Nacional de Matemática Aplicada e Computacional. Proceeding Series of the Brazilian Society of Computational and Applied Mathematics* (ZANOTELLI et al., 2017). Extends the robustness analysis of the fuzzy connectives based on the pointwise sensitivity of such operators. Starting with an evaluation of the sensitivity in fuzzy negations, triangular norms and conorms, we apply the results in the class of fuzzy difference operators and their dual construction. The paper formally states that the robustness preserves the projection functions related to fuzzy (co)difference operators.

9. Rosana Zanotelli, Renata Reiser, Adenauer Yamin and Benjamín Bedregal, Robustness of intuitionistic fuzzy difference operators, *Uncertainty Modelling in Knowledge Engineering and Decision Making: Proceedings of the 12th International FLINS Conference* (ZANOTELLI et al., 2016). Extends the robustness analysis of the fuzzy connectives based on the pointwise sensitivity of such operators. Starting with an evaluation of the δ sensitivity in representable fuzzy negations, triangular norms and conorms, we apply the results in the class of fuzzy difference operators and their dual construction. The paper formally states that the robustness preserves the projection functions related to intuitionistic fuzzy (co)difference operators.

10. Rosana Zanotelli, Wilson Cardoso and Renata Reiser, Robustez de Operadores Diferença Fuzzy Intuicionista, *XX Encontro de Pós-Graduação-Anais 2016*, url:https://wp.ufpel.edu.br/enpos/anais/anais2016. Extends the study of robustness in fuzzy systems. We consider the robustness analysis defined by the δ -sensibility of the operators of (co) difference in the Atanassov Intuitionist Fuzzy Logic.

11. Rosana Zanotelli, Renata Reiser, Simone Costa, Luciana Foss and Benjamín Bedregal, Towards robustness and duality analysis of intuitionistic fuzzy aggregations, *2015 IEEE International Conference on Fuzzy Systems (FUZZ-IEEE)* (ZANOTELLI et al., 2015). Studies the robustness of intuitionistic fuzzy connectives in fuzzy reasoning. Starting with an evaluation of the sensitivity in representable fuzzy negations, we apply the results in the intuitionistic (S, N)-implication class and its dual construction. The paper formally states that the robustness preserves the projection functions in this class and corresponding dual operators.

12. Rosana Zanotelli, Renata Reiser, Simone Costa, Luciana Foss and Benjamín Bedregal, Robustness of intuitionistic fuzzy implications: the Yager's implication case study, *WEIT 2015 - III Workshop-Escola de Informática Teórica* (ZANOTELLI et al., 2015). Studies the robustness of intuitionistic fuzzy implications in fuzzy reasoning. Starting with an evaluation of the sensitivity in representable fuzzy negations, we apply the results in the intuitionistic fuzy implication class and its dual construction. The paper formally states that the robustness preserves the projection functions in the Yager's implication and dual operator.

8.2.3.3 Contributions reported in other conference proceedings joint with the LUPS and MFFMCC research groups

1. Correlation coefficient analysis based on fuzzy negations and representable automorphisms (BERTEI et al., 2016).

2. Correlation Analysis of Intuitionistic Fuzzy Connectives (BERTEI et al., 2017).

3. Atanassov's Intuitionistic Fuzzy Entropy: Conjugation and Duality (COSTA et al., 2016).

8.2.3.4 Awards

The following work was awarded as the second best work of the event:

- Rosana Zanotelli, Renata Reiser and Benjamin Bedregal, Towards Inference Schemes in Approximate Reasoning using n-Dimensional Fuzzy Logic, *Proceedings of WEIT 2019 - Workshop-Escola de Informática Teórica* (ZANOTELLI et al., 2019)

8.3 Further work

Further work is in progress in order to characterize other classes of n-dimensional fuzzy implications and correlate their properties with the following theoretical research topics:

- The research on the n-dimensional upper simplex, in a more comprehensive study, applying the methodology to new classes of fuzzy connectives;
- Extending the study of the most relevant axiomatizations, based on recent results to establish the desirable properties of n-dimensional fuzzy implications and exploring applications of such operators;
- Exploring the implication residuum principle properties which are here extensively studied, leading to the characterization of new families of n-dimensional implications of left-continuous n-dimensional t-norms on $L_n(U)$, and also considering their application in the AR models based on inference schemes of deductive systems;
- Construction of admissible linear orders providing new analysis of main properties of fuzzy implications on $L_n(U)$ and enabling the comparison of multi-data of applications based on multiple alternatives/attributes and related to the MCDC research area.

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