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Dissertation

Generalized Atanassov's Interval-valued Intuitionistic Fuzzy Index - Construction of Atanassov's Interval-valued Fuzzy Entropy from Interval-Valued Fuzzy Implication Operators

Lidiane Costa da Silva

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Advisor: Prof[a]. Dr[a]. Renata Hax Sander Reiser
Coadvisor: Prof. Dr. Adenauer Corrêa Yamin

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ABSTRACT

SILVA, Lidiane Costa da. **Generalized Atanassov's Interval-valued Intuitionistic Fuzzy Index - Construction of Atanassov's Interval-valued Fuzzy Entropy from Interval-Valued Fuzzy Implication Operators**. 2018. 83 f. Dissertation (Masters in Computer Science) – Post Graduate Program in Computation, Center of Tecnological Development, Universidade Federal de Pelotas, Pelotas, 2018.

Despite its widespread research in applied areas, recognized limitations of Fuzzy Logic (FL) justify the search for higher levels of abstraction, using extensions such as type-2 fuzzy logic (T2FL) for the representation of information in fuzzy reasoning systems. This proposal considers the two intersection areas of T2FL, the intuitionist fuzzy logic (A-IFL) and the interval-valued fuzzy logic (IvFL). The interval-valued intuitionist fuzzy logic introduced by Atanassov in 1969 (A-IvIFL) considers both the imprecision of data in the membership function and the hesitation in determining its complementary relation - the non-membership functions. In A-IvIFL approach, the principles of A-IFL are preserved and the forms of data representation are expanded, adding not only the hesitation information related to experts with respect to non necessarily complementary relations but also the imprecision information provided by the interval-valued intuitionistic fuzzy index. Firstly, we consider the study of main properties verified by the axiomatic concept of hesitation index names - the generalized Atanassov's intuitionistic fuzzy index (A-GIFIx), and corresponding constructive methodology based on fuzzy implications. And, just in such context, this work introduces an extension of methodology which is able to preserve properties by making use of dual and conjugate operators. In addition, as the main contribution, the generalized Atanassov's interval-valued intuitionist fuzzy index (A-GIvIFIx) is discussed, including its axiomatic concept and related constructive methodology characterized in terms of interval-valued fuzzy implications, preserving main properties of an A-GIFIx. This work also presents new ways of obtaining A-GIvIFIx via dual and conjugated constructions, by the action of negation operators and automorphisms, respectively. Among the several applications of A-GIvIFIx as similarity, correlation and distance measures, we introduce a methodology to obtain the entropy via A-GIvIFIx contributing with multi-attribute systems based on T2FL. The application of the concept of admissible linear orders makes the comparison between interval results possible.

Keywords: atanassov's interval-valued intuitionistic fuzzy logic; intuitionistic fuzzy index; fuzzy entropy; duality; conjugation

RESUMO

SILVA, Lidiane Costa da. **Generalized Atanassov's Interval-valued Intuitionistic Fuzzy Index - Construction of Atanassov's Interval-valued Fuzzy Entropy from Interval-Valued Fuzzy Implication Operators**. 2018. 83 f. Dissertation (Masters in Computer Science) – Post Graduate Program in Computation, Center of Tecnological Development, Universidade Federal de Pelotas, Pelotas, 2018.

Apesar de abrangente, reconhecidas limitações da Lógica Fuzzy justificam a busca de níveis mais elevados de abstração, utilizando extensões como a lógica fuzzy tipo-2 (T2FL) para a representação de informações em sistemas de raciocínio fuzzy. Esta proposta considera a interseção de duas classes da T2FL, a lógica fuzzy intuicionista e a lógica fuzzy valorada intervalarmente. A lógica fuzzy intuicionista valorada intervalarmente introduzida por Atanassov em 1969 considera tanto a imprecisão de dados na função de pertinência quanto a hesitação na determinação de sua relação complementar - a função de não-pertiência. Nesta abordagem, preservam-se os princípios da lógica fuzzy intuicionista e ampliam-se as formas de representação, agregando à informação da imprecisão provida pelo índice fuzzy intervalar também a informação da hesitação de especialistas quanto aos graus de pertinência em relações não necessariamente complementares. Considera-se primeiramente o estudo das propriedades verificadas pela axiomatização do conceito de índice de hesitação proposto por Atanassov - o índice fuzzy intuicionista generalizado (A-GIFIx), e a correspondente metodologia de construção baseada em implicações fuzzy. E, já neste contexto, o trabalho introduz uma discussão de propriedades preservadas por operadores duais e de conjugação e apresenta uma extensão da referida metodologia. E, como principal contribuição este trabalho introduz a generalização do índice fuzzy intuicionista valorado intervalarmente (A-GIvIFIx), sua conceituação axiomática, uma metodologia de construção caracterizada em termos de implicações fuzzy valorados intervalarmente e ainda estuda condições que garantam a preservação das principais propriedades do A-GIFIx pela ação operadores duais e de conjugação. Este trabalho apresenta ainda novas formas de obtenção do A-GIvIFIx via construções duais e conjugadas, pela ação de operadores de negação e de automorfismos, respectivamente. Dentre as diversas aplicações do A-GIvIFIx como relações de medidas de similaridade, correlação e distâncias, estes trabalho introduz novas metodologias de obtenção da entropia via A-GIvIFIx contribuindo com sistemas multi-atributos baseados em T2FL. A aplicação do conceito de ordens lineares admissíveis viabiliza a comparação entre resultados intervalares.

Palavras-Chave: lógica fuzzy intuicionista valorada intervalarmente; índice fuzzy intuicionista; entropia fuzzy; dualidade; conjugação

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LIST OF ABBREVIATIONS AND ACRONYMS

T2FL	Type-2 Fuzzy Logic
TFS	Theory of Fuzzy Sets
TCL	Theory of Classical Logic
FS	Fuzzy Set
IvFS	Interval-valued Fuzzy Set
A-IFS	Atanassov‘s Intuitionistic Fuzzy Set
A-IvIFS	Atanassov‘s Interval-valued Intuitionistic Fuzzy Set
FL	Fuzzy Logic
IvFL	Interval-valued Fuzzy Logic
A-IFL	Atanassov‘s Intuitionistic Fuzzy Logic
A-IvIFL	Atanassov‘s Interval-valued Intuitionistic Fuzzy Logic
SFN	Strong Fuzzy Negation
IFN	Intuitionistic Fuzzy Negation
SIFN	Strong Intuitionistic Fuzzy Negation
IvFN	Interval-valued Fuzzy Negation
IvIFN	Interval-valued Intuitionistic Fuzzy Negation
SIvIFN	Strong Interval-valued Intuitionistic Fuzzy Negation
IvFA	Interval-valued Fuzzy Automorphism
IvIFA	Interval-valued Intuitionistic Fuzzy Automorphism
IvT	Interval-valued Fuzzy T-norma
IvS	Interval-valued Fuzzy T-conorma
IFIx	Intuitionistic Fuzzy Index
A-GIFIx	Generalized Intuitionistic Fuzzy Index
A-GIvIFIx	Generalized Interval-valued Intuitionistic Fuzzy Index
A-IFE	Atanassov‘s Generalization Intuitionistic Fuzzy Entropy
A-IvFE	Atanassov‘s Generalization Interval-valued Intuitionistic Fuzzy Entropy

CONTENTS

1 INTRODUCTION

Despite its widespread research in applied areas, recognized limitations of Fuzzy Logic (FL) justify the search for higher levels of abstraction, using extensions such as type-2 fuzzy logic (T2FL) for the representation of information in fuzzy reasoning systems.

This proposal considers the intersection of two relevant areas of T2FL, the intuitionist fuzzy logic (A-IFL) (ATANASSOV, 1986) and the interval-valued fuzzy logic (IvFL) (MOORE, 1962). The interval-valued intuitionist fuzzy logic introduced by Krassimir T. Atanassov in 1969 (A-IvIFL) considers both the imprecision of data in the membership function and the hesitation in determining its complementary relation - the non-membership functions (ATANASSOV, 1999).

In A-IvIFL approach, the principles of A-IFL are preserved and the forms of data representation are expanded, adding not only the hesitation information related to experts with respect to non necessarily complementary relations but also the imprecision information provided by the interval-valued intuitionistic fuzzy index.

Firstly, we consider the study of the main properties verified by the axiomatic concept of hesitation intuitionistic index, named as the generalized Atanassov's intuitionistic fuzzy index (A-GIFIx), and corresponding constructive methodology based on fuzzy implications and involutive negations.

In such context, this work introduces an extension of this methodology which is able to preserve properties by making use of dual and conjugate operators.

In sequence, as the main contribution, the generalized Atanassov's interval-valued intuitionistic fuzzy index (A-GIvIFIx) proposed in this work, includes its axiomatic concept and related constructive methodology characterized in terms of interval-valued fuzzy implications and strong negations, preserving the main properties of an A-GIFIx (BARRENECHEA et al., 2009).

This work also presents a novel way of obtaining A-GIvIFIx via dual and conjugated constructions, by the action of negation operators and automorphisms, respectively.

Among the several applications of A-GIvIFIx such as similarity, correlation and distance measures, we introduce a methodology to obtain the entropy via A-GIvIFIx, contributing with multi-attribute systems based on A-IvIFL (LIN; XIA, 2006).

Since the new method to obtain fuzzy entropy is defined by an interval-valued, resulting from the aggregation of all values of the A-IvIFIx, this work needs to consider the concept of admissible linear orders, making the comparison of entropy data resulting from distinct A-IvIFSs possible (MIGUEL et al., 2016).

1.1 Relevance of the Atanassov's Intuitionistic Fuzzy Index

Atanassov's intuitionistic fuzzy Index (A-IFIx) is a powerfull framework modelling the degree of each element in a A-IFS based on the pair of the membership degree and the non-membership degree, which is obtained by subtracting the sum of membership and non-membership from one. In this sense, the subtraction should be positive and less than or equal to one.

Thus, the A-IFIx provides the measurement of the effect of working with A-IFSs. It is characterized as the main condition of intuitionism in A-ITSs, providing very valuable information of each element and taking advantage of this potentiality in different applications.

The treatment for the uncertainty or lack of information of an expert into assigning correct values in the choice of the membership functions is a widely used strategy contributing to obtain more reliable fuzzy systems.

So, by using the A-IFIx values for representing expert's uncertainty in determining the (non-)membership degree related to an A-IFS, we are able to model the degree of intuitionism (or the hesitancy degree) which the expert has in giving his evaluation. Moreover, such representation of the uncertainty of the expert in determining the value of the (non-)membership degree, is closely related to the interval-valued fuzzy sets theory.

Several applications of A-IFIx have been developed, with significative contribuitons in multi-attributed decison making based on fuzzy systems. Concepts as similarity, bissimilarity, correlation, entropy and distance measures are applied in many areas as edge detection image processing, segmentation, decision making, fault-tree analysis and pattern recognition (GEORGE KLIR, 1995).

The new concept of Generalized Atanassov's Intuitionistic Fuzzy Index extends the expression of an A-IFIx as conceived by Atanassov (BARRENECHEA et al., 2009). From this axiomatic definition of A-GIFIx, a construction method of A-IFIx and fuzzy entropy emerge as interesting characterizations, by using order automorphisms. This methdology is studied and extended in this work.

1.2 Main Proposals

The Atanassov's intuitionistic fuzzy index (A-GIvIFIx), frequently called as the degree of hesitancy or indeterminance of an element in an Atanassov-intuitionitic fuzzy set (A-IFS), provides a measure of the lack of information supporting a given incomplete/inconsistent information related to a (non)membership evaluation given by an expert. The modelling of these fuzzy systems are based on Atanassov's intuitionistic fuzzy logic (A-IFL).

In addition, when the membership degree of an element cannot be precisely defined or we can just provide bounds for it, the corresponding value is considered as (non)membership interval. And, we can make use of Interval-valued fuzzy logic (IvFL) in modelling of fuzzy systems.

Integrating both approches, we consider the Atanassov interval-valued intuitionistic fuzzy logic (A-IvIFL) as an increasingly popular extension of fuzzy set theory. Allowing both strategies, the expression related to the expert uncertainty in identifying a particular membership function and the possibility to approximate the (unknown) membership degrees.

Thus, these theoretical studies and formal results proposed in this work underlie applied researches based on fuzzy system in which either the hesitation of experts in precise knowledge or the unknown data related to membership and nonmemberhip degrees, are considered, which are represented by pairs of intervals in $U = [0, 1]$.

Making use of A-IvIFL, this paper integrates both approaches:

(i) a new concept of the Generalized Atanassov's Intuitionistic Fuzzy Index associated with a strong intuitionistic fuzzy negation is characterized in terms of fuzzy implication operators which is described by a construction method with automorphisms, considering the results in (BARRENECHEA et al., 2009) and (BUSTINCE et al., 2011);

(ii) in (BUSTINCE et al., 2011), by means of special aggregation functions applied to the A-GIFIx, the Atanassov's intuitionistic fuzzy entropy is introduced.

Following these researches, this work studies the interval extension of an A-GIFIx, by considering the concept of conjugate and dual fuzzy implications, mainly interested in the representation method (CORNELIS; DESCHRIJVER; KERRE, 2004) having the impact on many properties satisfied by the generated operations.

Additionally, A-GIvIFIx associated with the standard negation together with known fuzzy implications are considered: Lukaziewicz, Reichenbach, Gaines-Rescher (LIN; XIA, 2006).

1.3 Objectives

Focusing on entropy measures and closed related studies presented by Burillo and Bustince (BURILLO; BUSTINCE, 1996), this proposal aims to contribute with a different way to explore the interval-valued fuzzy set model, offering application developers a new method of construction of A-IvIFE from A-GIvIFIx.

Based on formal studies, this work can contribute for multi-criteria decision making problem, ranking the alternatives based on another way for the interval-valued fuzzy set model, offering application developers a new method of construction of entropy by the intuitionistic fuzzy index, that means, obtaining A-IvIFE from A-GIvIFIx via interval-valued fuzzy implications, interval-valued idempotent aggregation and involutive negation operators.

More specifically, the following partial objectives are considered in this work.

(i) Characterization of the state-of-the-art on A-IvIFL and revision of the main concepts of A-IvIFL and A-IFL, definitions of basic connectives such as fuzzy implications, fuzzy negations and aggregations focusing on their algebraic properties, dual and conjugate construction;

(ii) Revision of axiomatic definition of A-IFIx in the sense of A-GIFIx (BUSTINCE; BARRENECHEA; MOHEDANO, 2004) including concepts and main properties of representable fuzzy connectives;

(iii) Study of IvFL, main properties of interval-valued fuzzy connectives focussing on the class of representable fuzzy (co)implications, including conjugate and dual operators;

(iv) Introduction of the axiomatic definition of the A-GIvIFIx in terms of conjugated function using automorphisms and also analysing properties of dual functions associated with the class of interval fuzzy (co)implications generated by idempotent interval aggregations;

(v) Analysis of truly intuitionist properties of interval-valued fuzzy (co)implications connecting to A-GIvIFIx, as interval extensions of the A-GIFIx approach;

(vi) Revision of wide concepts of A-IvIFL used to study entropy measures, in terms of aggregation operators and A-GIvIFIx;

(vii) Proposal of axiomatic definition of a constructive method to obtain entropy in terms of A-IvIFIx obtained by interval intuitionist fuzzy implications, idempotent aggregation and pairs of mutual dual functions;

(viii) Description of exemplification and discussion of possible applications of entropy obtained by the constructed methodology based on aggregation of A-GIvIFIx.

1.4 Context from Logical Approaches

In this section, we present an overview of the main contributions of the fuzzy logic and two multi-valued fuzzy logic approaches, considered relevant to achieve the objectives of this work. As extensions of fuzzy logic, we consider the interval valued intuitionistic fuzzy logic and, in an more general sense, the type-2 fuzzy logic.

1.4.1 Fuzzy Logic

The Theory of Fuzzy Sets (TFS) was formalized by the mathematician Lofti Asker Zadeh (ZADEH, 1965) by extending the concepts of Theory of Classical Logic (TCL), characterizing the attribution of membership degrees to the elements of a fuzzy set mainly depending on application contexts. Technological resources based on Boolean logic were not enough to automate industrial activities or even to compute with uncertainty of real problems (FODOR; ROUBENS, 1994).

The logical approach based on Fuzzy Logic in industrial applications happened with greater importance in Europe and after 1980 in Japan, highlighting the application of fuzzy systems in the Fuji Electric Company and later Hitachi Company, developing a metro control system based on fuzzy logic. And, around 1990 that fuzzy logic arised as a greater interest in companies from the United States (GEORGE KLIR, 1995).

The main advantage associated with the development systems based on TFS is to obtain a powerful mathematical model which not only is able to interpret the uncertainty of linguistic terms from natural language but also makes it possible to produce calculations even when we deal with inaccurate information in computer programming languages.

Due to the development of numerous practical possibilities and theoretical foundation together with the great commercial success of its applications, FL is considered nowadays as fundamental theory for a logical approach, modeling uncertainty with wide acceptance in relevant research areas such as artificial intelligence, natural language, expert systems, neural networks, control theory and decision making for computational processes.

Technological resources whose modeling consider uncertainty in their specification become logically specified from the use TFS, promoting credibility and reliability since planning to optimization in the development of such systems. For instance, by considering the development of electronic components, such as elevator control (Hitachi, Fujitech and Mitsubishi), signal analysis applied in TV image adjustment (Sony), camera autofocus video (Canon), video image stabilizer (Panasonic) and even fraud detection on credit cards. (GEORGE KLIR, 1995)

1.4.2 Interval-valued Intuitionistic Fuzzy Logic

Atanassov and Gargov (ATANASSOV; GARGOV, 1989) propose the interval-valued intuitionistic fuzzy logic based on the notion of interval-valued intuitionistic fuzzy sets, addressing a mathematical and more intuitive method than classical logic, which is able to consider ambiguous or uncertainty, easily integrated with imprecise information.

A-IvIFL not only deals with the indecision inherent in natural language variables modelling computer systems, but also collaborates with two other interpretations:

(i) Firstly, the interpretation which is achieved when intervals may be considered as particular types of fuzzy sets, representing the imprecision of a variable depending on the computational context; and

(ii) secondly, the indecision about the relation between membership and non-membership degrees, not necessarily related as complementary degrees.

In this context, the former is concerned with calculations and numerical errors, by regarding information related to various experts demanding membership and non membership degrees. The latter is consistent to non-zero intuitionistic fuzzy index.

A-IvIFL reinforces these interpretations by using the duality principle and aggregation operators, providing a more flexible modelling for the truth associated with each variable. Thus, a more realistic analysis of the veracity of this variable can be inferred as much as from its interval membership degree and, from the complement, its relation with its interval nonmembership degree.

Applications involving systems based on A-IvIFL together with computational tools, such as neural networks and evolutionary programming, expert systems, approximation reasoning and digital image processing. Also noteworthy are applications in resource management, military strategies, medical diagnosis, pattern recognition and clustering analysis in logistics (BUSTINCE; BARRENECHEA; MOHEDANO, 2004).

However, despite the relevant advances, there is no consensus in order to consolidate a solution to guide theoretical basis as well as the mathematical methods to support the area of decision making based on multiple attributes.

This convergence is still a great research challenge, justified by many factors, among which the following stand out:

(i) insufficient knowledge of decision-makers;

(ii) the ever-increasing need to aggregate two or more possible judgments;

(iii) the challenging ability to deal with subjective characteristics of alternatives of fuzzy preference modelling supporting multi-atributes in decision making.

All these factors generate uncertain information that must be mapped from the modelling of decision-making systems based on multiple attributes (DUBOIS; PRADE, 2000).

In order to compare data in this work, some results are focussed on the study of A-GIvIFIx to obtain entropy, which is close to analyse other parameters as distance, similarity, bissimilarity, correlation, acuracy, score, and many other ones performed over A-IvIFSs. Some authors put forward their axiomatic definitions in constructive methods to obtain interval-valued entropies for A-IvIFSs, including distance or similarity measures.

1.4.3 Type-2 Fuzzy Logic

As a special class of extended fuzzy sets introduced by Zadeh (ZADEH, 1975) and Sambuc (SAMBUC, 1975), T2FSs were mathematically defined by Mendel and Karnik in 1998 (KARNIK; MENDEL, 1998) including the study of first operations on such sets. Moreover, many other relevant works in IvFSs were preliminarily studied by Dubois and Prade's (DUBOIS; PRADE, 1991).

In this new and comprehensive approach of T2FSs, the uncertainty about the membership function is defined by a fuzzy set of type-1, collaborating more significantly in the modeling of fuzzy systems involving the approximation of random data in temporal evolution (MENDEL, 2003).

Such approach contributes to the generalization of the fuzzy set theory, since if there is no uncertainty in the membership function, then a fuzzy set of type-2 is reduced to a fuzzy set of type-1 (DUBOIS; PRADE, 1991).

An application based on type-2 fuzzy logic (DESCHRIJVER; KERRE, 2005) increases the contribution to identify models or prediction of behaviour from expert information. Fuzzy systems based on T2FL are fuzzy systems in which at least one of their antecedent or consequent fuzzy sets are fuzzy sets of type-2 (KARNIK; MENDEL, 2001).

The logical approach based on T2FSs is applied in many technological applications and interesting examples follow from (SANCHEZ; CASTILLO; CASTRO, 2015; MIGUEL et al., 2017) exploring the use of T2FL to control a mobile robot, create a hybrid system using T2FL for the prediction of the survival time of myeloma patients, formalize an expert system for the realization of Shopping via web including application for analysis and estimation of the survival time of wireless sensor networks.

In this new and comprehensive approach of T2FSs, the uncertainty about the membership function is defined by a type-1-fuzzy set, collaborating more significantly in the modeling of fuzzy systems involving the approximation of random data in temporal evolution (MENDEL, 2003).

1.5 Work Outline

This work is organized in eight chapters, summarized below.

Chapter 1 is the introduction, where the relevance of the research theme is presented together with the context from logical approach, main contributions and objectives are also registered. It is followed by description of related work, in Chapter 2.

Chapter 3 describes the basics concepts of Atanassov's intuitionistic fuzzy logic as well as dual operators, conjugation and fuzzy connectives, aggregators, implications. In addition, the main concepts of Atanassov's intuitionistic fuzzy logic and order relation are reported, including the Atanassov's intuitionistic fuzzy index definition.

In Chapter 4 the generalized Atanassov's intuitionistic fuzzy index is reported, with its main axioms, obtaining dual and conjugate connectives in a special class of implications. In this chapter we also present the concepts and properties of Atanassov's generalized intuitionistic fuzzy entropy, obtained through of Atanassov's generalized intuitionistic fuzzy index considering a case study.

In Chapter 5, are presented the main concepts of Interval-valued fuzzy logic, describing its connectives, relations of order and in the sequence the interval intuitionistic fuzzy logic is also described from its main connectives.

The Chapter 6 presents the relation of the interval extension of generalized intuitionistic fuzzy index and relations with interval-valued fuzzy connectives.

The Chapter 7, presents the main relationship of intuitionistic index its conjugate, dual operators with the interval-valued intuitionistic fuzzy entropy, preserving fuzzyness and intuitionism.

In Chapter 8 the main contributions of this work, as well as the possibilities of continuity of activities.

2 RELATED WORKS

In this section we reported reviewed works related to our proposal in order to introduce the interval extension of generalized Atanassov's intuitionistic fuzzy index, and discussed how our extension advances the state of the art.

The literature revision is focussed on three main aspects:

(1) Study of interval-valued intuitionistic fuzzy logic, considering important aspects to express interval extensions of generalized Atanassov's intuitionistic fuzzy index:

1. We review formal concepts of generalized Atanassov's intuitionistic fuzzy index that have been developed in the literature, as well as we study proposed notions related to interval-valued intuitionistic fuzzy logic, mainly interested in the results presented in (BARRENECHEA et al., 2009) and (BUSTINCE et al., 2011).
2. The previous work of Pankowska and Wygralak (PANKOWSKA; WYGRALAK, 2006) it is proposed a generalization of the intuitionistic index based on strong negations and triangular norms. This approach is used to construct flexible algorithms of group decision making.
3. In (BARRENECHEA et al., 2009) the Generalized Atanassov's Intuitionistic Fuzzy Index is characterized in terms of fuzzy implication operators and it is studied some special properties of the generalized Atanassov's intuitionistic fuzzy index. The new concept of A-GIFIx generalizes the expression given by Atanassov and a characterization method by means of fuzzy implication operators is constructed allowing presentation of simple expressions.
4. Different construction methods of Atanassov's intuitionistic fuzzy entropy (BURILLO; BUSTINCE, 1996) by means of special aggregation functions applied on generalized Atanassov's intuitionistic fuzzy index is presented.

(2) Study of fuzzy entropy obtained by interval extension and generalized expression of Atanassov's intuitionistic fuzzy index:

1. We survey the concept of entropy for intuitionistic fuzzy sets obtained by application of intuitionistic fuzzy index, considering relevants concepts presented in (BARRENECHEA et al., 2014; JING; MIN, 2013).
2. In this sense, starting with the concept of fuzzy entropy we extend this study to deal with entropy of Atanassov's interval-valued intuitionistic fuzzy sets and we also investigate such concept in the field of T2FSs (MIGUEL et al., 2017).

(3) Study of admissible linear orders and their application to compare interval-valued Atanassov intuitionistic fuzzy sets:

1. In (MIGUEL et al., 2016), interval-valued Atanassov intuitionistic OWA aggregations are studied using admissible linear orders and their application to decision making. Choquet integrals for aggregating information is represented using interval-valued Atanassov intuitionistic fuzzy sets. Algorithms are presented to choose the best alternative in a decision making problem.
2. This revision outlines the main problem to compare and consequently, to choose an appropriate total order for applications making use of interval extensions of intuitionistic fuzzy sets.
3. The study of linear orders for intervals by means of aggregation functions is presented in (BUSTINCE et al., 2013). The concept of an admissible order as a total order that extends the usual partial order between intervals in the proposed method to build these admissible orders are defined in terms of two aggregation functions, showing that some of the most used examples of total orders appearing in the literature are specific cases of such construction.

For that, we searched in previous works containing structured test as (i) (ZHANG, 2013), reporting theories and methods under fuzzy logic and aggregation operator theory, and (ii) (BUSTINCE et al., 2016), providing a historical account of type-2 fuzzy sets and their relationships. Both research revisions were carried out using systematic literature review.

As supporting our work, the relevant research presenting results from Atanassov's interval-valued intuitionistic fuzzy Logic are briefly described in Table 1.

Analogously, papers helping in the consolidation of the constructive method to obtain interval-valued fuzzy entropy from Atanasov's interval-valued intuitionistic fuzzy values are listed in Table 2.

Table 1 – Related Works in Interval-valued Intuitionistic Fuzzy Logic

Authors	**Study Contribution**
(DUGENCI, 2016)	***A-IvIFS*** *Proposed the new generalized distance measure and interval valued intuitionistic fuzzy sets for solving problems related to group decision making.*
(CHEN; TSAI, 2016)	***IVIFWGA*** *Proposed a novel method based on interval-valued intuitionistic fuzzy ordered weighted geometric averaging and interval-valued intuitionistic fuzzy hybrid geometric averaging operators.*
(DYMOVA; SEVASTJANOV, 2016)	***IVIFV*** *Proposed interval-valued intuitionistic fuzzy values based on Dempster–Shafer theory due to some limitations and drawbacks of previous studies.*
(WAN et al., 2015)	***AIVIFV*** *Developed a novel model for solving problems by using problems with incomplete attribute weight information and AIVIFVs.*
(BARRENECHEA et al., 2014)	***IVFRs*** *Proposed general algorithm to solve problems by using IVFRs.*
(XU; SHEN, 2014)	***AIVIFN*** *Extended the method to take account of the DMs‘ assessment information.*
(CHEN, 2014)	***IVIF*** *Presented a new IVIF prioritized aggregation operator to aggregate the IVIF ratings of the alternatives.*
(WAN et al., 2016)	***TIFN*** *Extended some operators including TIFOWA, TIFOWG, IFHWA, TIFGOWA, and TIFGHWA based and multi-objective programming.*
(WANG et al., 2016)	*Combined the unreliable evidence sources in MCDM method in intuitionistic fuzzy environment.*

Table 2 – Related Works Interval-valued Intuitionistic Fuzzy Entropy

Authors	Study Contribution
(ZENG; LI, 2006)	*Discussed the relationship between the similarity measure and entropy in IVFSs and proved that similarity measure and entropy of IVFSs can be transformed by each other.*
(HUNG; YANG, 2008)	*Constructed their axiom definition of entropy of IFSs and proposed two families of entropy measures.*
(VLACHOS; SERGIADIS, 2007)	*Introduced information-theoretic discrimination measures and cross-entropy for IFSs and derived an extension of the DeLuca-Termini non-probabilistic entropy for IFSs.*
(QIAN-SHENG; JIANG, 2008)	*Put forward a nonprobabilistic entropy of vague set by means of the intersection and union of membership and nonmembership degrees.*
(ZHANG; ZHANG; MEI, 2009)	*Proposed a new axiomatic definition of entropy of IVFSs based on distance and investigated the relationship between entropy and similarity measure of IVFSs.*
(BURILLO; BUSTINCE, 1996)	*Defined the distance measure between IFSs and gave a definition of entropy for IFSs.*
(SZMIDT; KACPRZYK, 2001)	*Supplied a geometric interpretation of IFSs and proposed another entropy measure with a ratio of distances.*

3 ATANASSOV'S INTUTIONISTIC FUZZY LOGIC

Atanassov's Intutionistic Fuzzy Logic is conceived as multi-valued fuzzy logic extending main concepts of fuzzy logic which is based on the theory of Atanassov's Intuitionistic Fuzzy Sets.

3.1 Basic Concepts of Fuzzy Logic

Introduced by Zadeh in 1965 (ZADEH, 1965), Fuzzy Logic is non-classical logic capable of numerically modeling ambiguous, uncertain or vague information, described through a natural language aiding the modeling of the human ability to make decisions from information obtained by expert systems (ROSS, 2004).

In classical set theory, an element belongs to or does not belong to a given set, however, there are cases where the pertinence between elements and sets is not precise, and it is not possible to discreetly define whether an element belongs or not to a set. Systems that model uncertainties, for example, do not always have well defined pertinence boundaries (SILER; BUCKLEY, 2004; CARLSSON; FULLER, 2002).

In the theory of fuzzy sets, the relevance of an element to a given set is relaxed; an element may partially belong to a particular set, rather than simply belong or not belong to a set. Thus, the membership degree of a given element in a fuzzy set is given by a membership function, considering a universe of discourse $\chi \neq \emptyset$.

According with (ZADEH, 1965), a fuzzy set A is characterized by its membership function $\mu_A : \chi \to U$ and $\mu_A(x)$ interpreting the membership degree of an element x in fuzzy set A. In this sense, a fuzzy set A can be decribed as a set of ordered pairs, where each generic element x is associated with its degree of relevance $\mu_A(x)$:

$$A = \{(x, \mu_A(x)) : x \in \chi,\ \mu_A(x) \in [0,1]\}. \tag{1}$$

Regarding the form of the membership functions, it is restricted to a certain class of functions, represented by some specific parameters. The most common forms are: linear by parts (triangular, trapezoidal), Gaussian, sigmoid and singleton (unitary sets) (ROSS, 2004).

By considering the natural order $\leq$ on U, the lattice $L(U) = (U, \leq, \vee, \wedge, 1, 0)$ has the supremum and infimum operations both given as the following

$$x \vee y = \max(x, y) \quad \text{and} \quad x \wedge y = \min(x, y). \tag{2}$$

3.1.1 Conjugation Operators

Automorphisms are considered as a generation of new connectives, preserving as algebraic properties the classes of these logical connectives.

According with (KLEMENT; NAVARA, 1999, Def. 4.1), an automorphism $\phi : U \to U$ is a bijective, strictly increasing function satisfying the monotonicity property:

A1: $x \leq y$ if and only if $\phi(x) \leq \phi(y)$, $\forall x, y \in U$.

In (BUSTINCE; BURILLO; SORIA, 2003), $\phi : U \to U$ is a function satisfying the continuity property and the boundary conditions:

A2: $\phi(0) = 0$ and $\phi(1) = 1$.

The set $Aut(U)$ of all automorphisms are closed under composition:

A3: $\phi \circ \phi' \in Aut(U)$, $\forall \phi, \phi' \in Aut(U)$.

In addition, there exists the inverse $\phi^{-1} \in U$, such that

A4: $\phi \circ \phi^{-1} = id_U$, $\forall \phi \in Aut(U)$.

Thus, $(Aut(U), \circ)$ is a group with the identity function being the neutral element.

The action of an automorphism $\phi : U \to U$ on a function $f : U^n \to U$ is called the **conjugate of** f and given by the following expression:

$$f^{\phi}(x_1, \ldots, x_n) = \phi^{-1}(f(\phi(x_1), \ldots, \phi(x_n))). \tag{3}$$

Example 1. *For all $k, l \in \{1, .., n\}$, let $\phi_k, \psi_{k,l}$ be functions in $Aut(U)$ given by:*

$$\psi_{k,1}(x) = x^{\frac{l}{k}} \qquad \psi_{k,l}^{-1}(x) = \sqrt[l]{x^k} \tag{4}$$

$$\phi_k(x) = \frac{(kx+1)^2 - 1}{k(k+2)} \qquad \phi_k^{-1}(x) = \frac{\sqrt{(k^2+2k)x+1} - 1}{k} \tag{5}$$

Both results can be easily observed:

- *By taking $l = 1$ in Eq.(4), we obtain that $\psi_k(x) = x^k$ and $\psi_k^{-1}(x) = \sqrt[k]{x}$.*
- *And, when $k = 1$ in Eq.(5), $\phi(x) = \frac{(x+1)^2-1}{3}$ and $\phi^{-1}(x) = \sqrt{3x+1} - 1$.*

3.1.2 Dual Operators

A function $N : U \to U$ is a *fuzzy negation* (FN) if

N1: $N(0) = 1$ and $N(1) = 0$;

N2: If $x \geq y$ then $N(x) \leq N(y)$, $\forall x, y \in U$.

Fuzzy negations satisfying the involutive property below are called *strong* fuzzy negations (BUSTINCE; BURILLO; SORIA, 2003):

N3: $N(N(x)) = x$, $\forall x \in U$.

Let N be a fuzzy negation and $f : U^n \leftrightarrow U$ be a real function. The N**-dual function of** f is denoted by $f_N : U^n \leftrightarrow U$ and defined as follows:

$$f_N(x_1, \ldots, x_n) = N(f(N(x_1), \ldots, N(x_n))). \tag{6}$$

Example 2. *For all $k, n \in \{1, 2 \ldots, n\}$, let $N^*, N_k, C_k : U \to U$ be strong fuzzy negations given by the corresponding expressions:*

$$N^*(x) = \frac{1-x}{1+x} \tag{7}$$
$$C_n^k(x) = \sqrt[n-k+1]{1 - x^{n-k+1}}. \tag{8}$$

In particular, based on (GEORGE KLIR, 1995, Theorem 3.4), every continuous fuzzy negation has a unique equilibrium point. Thus, the following holds:

(i) *when $n = 1$ in Eq.(8), we obtain $C^k(x) = 1 - x^k$;*

(ii) *when $k = 1$ in Eq.(8), we have the negation $C_n(x) = \sqrt[n]{1 - x^n}$ which has $e = \sqrt[n]{\frac{1}{2}}$ as the equilibrium point, meaning that $C_n(e) = e$.*

(iii) *And concluding, when $k = n = 1$ in Eq.(8), we obtain the standard fuzzy negation given as follows:*

$$N_S(x) = 1 - x; \tag{9}$$

3.1.3 Aggregation Operators

Fuzzy set theory and aggregation operators have become powerful tools to deal with decision making theories. Methods under fuzzy aggregation operator have been proposed and developed for effectively solving the decision making problems and numerous theoretical results and applications have been reported in the literature.

Among several definitions, see (TORRA, 2005) and (BUSTINCE; BARRENECHEA; MOHEDANO, 2004, Definition 2), an aggregation is a function $A : U^n \to U$ demanding, for all $\vec{x}, \vec{y} \in U^n$, the following conditions:

Ag1: $A(\vec{0}) = A(0, 0, \ldots, 0) = 0$ and $A(\vec{1}) = A(1, 1, \ldots 1) = 1$;

Ag2: If $\vec{x} = (x_1, x_2, \ldots, x_n) \leq \vec{y} = (y_1, y_2, \ldots, y_n)$ then $A(\vec{x}) \leq A(\vec{y})$;

Ag3: $A(\overrightarrow{x_\sigma}) = A(x_{\sigma_1}, x_{\sigma_2}, \ldots, x_{\sigma_n}) = A(x_1, x_2, \ldots, x_n) = A(\vec{x})$.

Ag4: $A(x, x, \ldots, x) = x, \forall x \in U$;

The process of aggregation combines several numerical values into a single value that somehow represents all the others. Thus, an aggregation is a function non-decreasing, commutative, and further, preserves the boundary conditions relating to the ends of the unitary interval (DESCHRIJVER; KERRE, 2005, Definition 4.1).

3.1.3.1 Ordered Weighted Averaging Operator (OWA)

The aggregation function *Ordered Weighted Averaging Operator*(OWA) was introduced by Yager (YAGER, 1988) providing a mean of aggregating values associated with satisfying multiple criteria. Thus, an OWA operator unifies both element behaviors into fuzzy sets, the conjunctive and the disjunctive.

An operator $OWA \colon U^n \to U$ is defined by the expression:

$$OWA(x_1, x_2, ..., x_n) = \sum_{j=1}^{n} w_j x_{\sigma(j)}, \forall x_1, x_2, ..., x_n \in U, \tag{10}$$

where $\sigma \colon N_n \to N_n$, is a σ-ordering permutation with non-negatives weight-parameters w_i non-negatives verifying the following conditions:

$$x_{\sigma(1)} \leq x_{\sigma(2)} \leq \ldots \leq x_{\sigma(n)} \quad \text{and} \quad \sum_{i=1}^{n} w_i = 1, \forall\ 0 \leq w_i \leq 1.$$

According to Yager and Kacprzyk (YAGER; KACPRZYK, 2012), OWA operators are aggregation functions which are commutative, idemponent, and have a compensatory behavior also satisfying properties **Ag1**, **Ag2** and **Ag3**.

In order to obtain distinct operators, one simply has to choose a particular value of weight w_i. Thus, parameterized families of aggregation operators can be defined from an OWA operator, including among many others, the minimum $(\min)$, the maximum $(\max)$, the median (M_d), the weighted mean (M_w) and the arithmetic mean (M). See these examples expressed in accordance with Table 3.

3.1.3.2 Disjuntive and Conjuntive Operators

According with (LESKI, 2003) definition, a triangular (co)norm is a binary aggregation $T(S) : U^2 \to U$ which is symmetric, associative, monotonic and has the

Table 3 – Aggregations obtained from the OWA operator

OWA Parameters	**Algebric Expression**
$\begin{cases} w_1 = 1 \\ w_i = 0 \text{ if } i \neq 1 \end{cases}$	$\min(x_1, x_2, \ldots, x_n) = x_1 \wedge x_2 \wedge \ldots \wedge x_n$
$\begin{cases} w_n = 1 \\ w_i = 0 \text{ if } i \neq n \end{cases}$	$\max(x_1, x_2, \ldots, x_n) = x_1 \vee \ldots \vee x_n$
$\begin{cases} w_{\frac{n+1}{2}} = 1 \text{ if } n \text{ is odd} \\ w_{\frac{n}{2}} = \frac{1}{2} \ ; w_{\frac{n}{2}+1} = \frac{1}{2} \text{ if } n \text{ is even} \\ w_i = 0 \text{ otherwise} \end{cases}$	$M_d(x_1, \ldots, x_n) = \begin{cases} x_{\sigma(\frac{n+1}{2})}, \text{if n is odd;} \\ \frac{1}{2}\left(x_{\sigma(\frac{n}{2})} + x_{\sigma(\frac{n}{2}+1)}\right), \text{if n is even.} \end{cases}$
$\sum_{i=1}^{n} w_i = 1, \forall w_i \in \mathbb{Q}^+$	$M_{w_1,\ldots,w_n}(x_1, \ldots, x_n) = \sum_{i=1}^{n}(w_i.x_i)$
$w_i = \frac{1}{n}, \forall i.$	$M(x_1, \ldots, x_n) = \frac{1}{n}\sum_{i=1}^{n} x_i = \sum_{i=1}^{n}\left(\frac{1}{n} \cdot x_i\right)$

neutral element 1 (0). This also means that, for all $x, y, z, t \in U$, the corresponding algebraic properties are verified:

T1: $T(x, y) = T(y, x)$;

T2: $T(x, T(y, z)) = T(T(x, y), z)$;

T3: $T(x, y) \leq T(z, t)$, if $x \leq z$ and $y \leq t$

T4: $T(x, 1) = x$;

S1: $S(x, y) = S(y, x)$;

S2: $S(x, S(y, z)) = S(S(x, y), z)$;

S3: $S(x, y) \leq S(z, t)$ if $x \leq t$ and $y \leq z$;

S4: $S(x, 0) = x$

In the following, by (KLEMENT; MESIAR; PAP, 1999), the expression of an N-dual operator of a triangular (co)norm is considered.

A function $T_N(S_N) : U^2 \to U$ is a t-conorm (t-norm) if, and only if, there exists a t-norm T (t-conorm S) such that for all $x, y \in U$, the following holds:

$$T_N(x, y) = N(T(N(x), N(y))), \quad (11)$$

$$S_N(x, y) = N(S(N(x), N(y))). \quad (12)$$

A t-conorm T_N given by Eq. (11) is called the t-conorm derived from T by the duality relation and, similarly a t-norm S_N given by Eq. (12) is called the t-norm derived from S by the duality relation, both defined with respect to the fuzzy negation N. When N is a strong fuzzy negation, then (T, T_N) ((S, S_N)) is a pair of mutual N-dual functions.

Table 4 shows examples of pairs of mutual dual t-norms and t-conorms based on previous results from (DUBOIS; PRADE, 2000).

Table 4 – Examples of (co)norms fuzzy triangles

Operators	Algebric Expression
Standard Intersection:	$T_M(x,y) = min\ \{x,y\}$
Standard Unity:	$S_M(x,y) = max\ \{x,y\}$
Algebraic Product:	$T_P(x,y) = x.y$
Probabilistic Sum:	$S_P(x,y) = x + y - xy$
Drastic Intersection:	$T_D(x,y) = \begin{cases} 0, & \text{if } x < 1, y < 1 \\ \min\{x,y\}, & \text{otherwise} \end{cases}$
Drastic Unity:	$S_D(x,y) = \begin{cases} 1, & \text{if } 0 < x \text{ and } 0 < y \\ \max\{x,y\}, & \text{otherwise} \end{cases}$
Lukasiewicz:	$T_L(x,y) = \max\{x + y - 1, 0\}$
Lukasiewicz:	$S_L(x,y) = \min\{x + y, 1\}$
Mimum Nilpotent:	$T_{nM}(x,y) = \begin{cases} 0, & \text{if } x + y \leq 1 \\ \min\{x,y\}, & \text{otherwise} \end{cases}$
Maximum Nilpotent:	$S_{nM}(x,y) = \begin{cases} 1, & \text{if } x + y \geq 1 \\ \max\{x,y\}, & \text{otherwise} \end{cases}$

3.1.4 (Co)Implications Operators

Fuzzy implications play an important role in Fuzzy Logic. In a broad sense, it is frequently applied to fuzzy control, analysis of vagueness in natural language and techniques of soft-computing as well as in the narrow sense, contributing to a branch of many-valued logic enabling the investigation of deep logical questions (BACZYNSKI; JAYARAM, 2007; BUSTINCE; BURILLO; SORIA, 2003; FODOR; ROUBENS, 1994).

A **fuzzy (co)implicator** $I(J) : U^2 \to U$ is a function verifying boundary conditions:

I0:$I(0,0) = I(0,1) = I(1,1) = 1$; **J0**: $J(0,0) = J(0,1) = J(1,1) = 0$.

Based on concepts introduced in (FODOR; ROUBENS, 1994), a **fuzzy (co)implication** $I(J) : U^2 \to U$ is a function verifying the following properties:

I1: If $x \leq z$ then $I(x,y) \geq I(z,y)$; **J1**: If $x \leq z$ then $J(x,y) \geq J(z,y)$;

I2: If $y \leq z$ then $I(x,y) \leq I(x,z)$; **J2**: If $y \leq z$ then $J(x,y) \leq J(x,z)$;

I3: $I(0,x) = 1$; **J3**: $J(1,x) = 0$

I4: $I(x,1) = 1$; **J4**: $J(x,0) = 0$

I5: $I(1,0) = 0$; **J5**: $J(1,0) = 1$.

Several reasonable properties may be required for fuzzy (co)implications:

I6: $I(1, x) = x$;
I7: $I(x, I(y, z)) = I(y, I(x, z))$;
I8: $I(x, y) = 1 \Leftrightarrow x \leq y$;
I9: $I(x, y) = I(N(y), N(x))$, N is a SFN;
I10: $I(x, y) = 0 \Leftrightarrow x = 1$ and $y = 0$;

J6: $J(0, x) = x$;
J7: $J(x, J(y, z)) = J(y, J(x, z))$;
J8: $J(x, y) = 0 \Leftrightarrow x \geq y$;
J9: $J(x, y) = J(N(y), N(x))$, N is a SFN;
J10: $J(x, y) = 1 \Leftrightarrow x = 0$ and $y = 1$.

Main results summarized in (BACZYNSKI; JAYARAM, 2007, Lemma 2.1) provide the structure to define fuzzy negations induced by fuzzy (co)implicators:

A function $I(J) : U^2 \to U$ satisfying **I0**(**J0**) and **I1**(**J1**) induces the definition of a fuzzy negation $N_I(N_J) : U \to U$ given as follows:

$$N_I(x) = I(x, 0) \quad \text{and} \quad N_J(x) = J(x, 1) \tag{13}$$

Moreover, main results presented in (REISER; BEDREGAL; BACZYNSKI, 2013, Proposition 4.3) provide the N-dual approach for fuzzy (co)implications.

Let N be a FN and (J) I be a (co)implication. Then (J_N) I_N defined according with Eq. (6) is a (implication) coimplication given as follows

$$I_N(x, y) = N(I(N(x), N(y))), \tag{14}$$

$$J_N(x, y) = N(J(N(x), N(y))). \tag{15}$$

Let $T(S)$ be a t-(co)norm and N be a FN. An $(S, N)-$implication ($(T, N)-$coimplication) is a fuzzy (co)implication $I_{S,N} : U^2 \to U$ defined by

$$I_{S,N}(x, y) = S(N(x), y) \tag{16}$$

$$J_{T,N}(x, y) = T(N(x), y). \tag{17}$$

In this work, we also consider the class of *S-implications* which is studied in (TRILLAS; VALVERDE, 1985, Theorem 3.2) also taking into account main concepts from (FODOR; ROUBENS, 1994, 10, Theorem 1.13) and introduced by Baczynsky and Jayaram in (BACZYNSKI; JAYARAM, 2007; BUSTINCE; BURILLO; SORIA, 2003).

Theorem 1. *(TRILLAS; VALVERDE, 1985, Theorem 3.2) Let N be a strong fuzzy negation. An implication $I : U^2 \to U$ is a* strong S-implication *if, and only if, it satisfies Properties **I1**, **I2**, **I6**, **I7**, and **I9**.*

Theorem 2. *(BACZYNSKI; JAYARAM, 2007, Theorem 1.6) Let N be a strong fuzzy negation. An implication $I : U^2 \to U$ is a* strong S-implication *if, and only if, it satisfies Properties **I1**, **I7** and N_I defined in Eq.(16) is a strong fuzzy negation.*

In the following, Table 5 reports the algebraic expressions of fuzzy implications considered in this work also including their corresponding coimplications, by taking the maximun and minimun operators. Each line in Table 5 is associated to a pair of mutual N_S-dual operators, meaning that Eq. (14) and (15) are both verified. In these cases, the operators are obtained considering extensions of the Lukaziewicz, Reichenbach, Klenee-Dienes, Gaines-Richard fuzzy (co)implications in order to preserve property **I8** and also including operator I_{30}.

Table 5 – Fuzzy Implications, coimplications and duality.

Fuzzy Implications	**Fuzzy Coimplications**
$I_{LK}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ 1-x+y, & \text{otherwise}; \end{cases}$	$J_{LK}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ y-x, & \text{otherwise}; \end{cases}$
$I_{KD}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ \max(1-x,y), & \text{otherwise}; \end{cases}$	$J_{KD}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ \min(1-x,y), & \text{otherwise}; \end{cases}$
$I_{RB}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ 1-x+xy, & \text{otherwise}; \end{cases}$	$J_{RB}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ y-xy, & \text{otherwise}; \end{cases}$
$I_{GR}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise}; \end{cases}$	$J_{GR}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ 1, & \text{otherwise}; \end{cases}$
$I_{30}(x,y)=\begin{cases} \max(1-x,y,0.5), & \text{if } 0<y<x<1, \\ \max(1-x,y), & \text{otherwise}; \end{cases}$	$J_{30}(x,y)=\begin{cases} \min(1-x,y,0.5), & \text{if } 0<x<y<1, \\ \min(1-x,y), & \text{otherwise}; \end{cases}$

3.2 Main Conceps of Intuitionistic Fuzzy Logic

The theory of intuitionistic fuzzy sets (ATANASSOV; GARGOV, 1989), extends the theory of fuzzy sets, associating to each element x in a universe $\mathcal{X} \neq \emptyset$, membership and non-membership degrees, both defined in the unit interval by the corresponding expressions ($\mu_A(x)$) and ($\nu_A(x)$), and such that the following natural relation is satisfied:

$$0 \leq \mu_A(x) + \nu_A(x) \leq 1. \tag{18}$$

Thus, expressions in Eq.(18) extend the fuzzy set theory, since membership and non-membership degrees are not necessarily complementary with respect to the unitary interval U.

Thus, an **intuitionistic fuzzy set** A consists into a set of pairs (μ_A, ν_A), whose components satisfy the natural restriction (ATANASSOV, 1986) given by Eq. (18). Therefore, it is assumed that an intuitionistic fuzzy set can be described as follows:

$$A = \{(x, (\mu_A(x), \nu_A(x))) : x \in \mathcal{X} \ e \ \mu_A(x) + \nu_A(x) \leq 1\},$$

where $\mu_A, \nu_A : \mathcal{X} \to U$ are the functions defining the corresponding **membership and non-membership degrees** of an element $x \in \mathcal{X}$ in A.

As a consequence, fuzzy set theory can be studied as a special case of intuitionistic fuzzy set theory, whose non-membership degree can be obtained through the equality:

$$\mu_A(x) + \nu_A(x) = 1.$$

In the modelling system of inference rules based on A-IFL, not only the membership function $\mu_A : \chi \to U$ is considered, but also the non-membership function $\nu_A : \chi \to U$. And, each element $\mathrm{x} \in \mathcal{X} \neq \emptyset$ is associated to a membership degree $\mu_A(\mathrm{x})$ and a non-membership degree $\nu_A(\mathrm{x}) = N_S(\mu_A(x))$, defining an intuitionistic fuzzy set (A-IFS) A, such that $0 \leq \mu_A(\mathrm{x}) + \nu_A(\mathrm{x}) \leq 1$.

Let $\tilde{U} = \{(x_1, x_2) \in U^2 \colon x_1 \leq N_S(x_2)\}$ be the **set of all intuitionistic fuzzy values** and $l_{\tilde{U}}$. The projection functions on $\tilde{U}$, $r_{\tilde{U}} : \tilde{U} \to U$ are given as follows:

$$l_{\tilde{U}}(\tilde{x}) = l_{\tilde{U}}(x_1, x_2) = x_1; \quad \text{and} \quad r_{\tilde{U}}(\tilde{x}) = r_{\tilde{U}}(x_1, x_2) = x_2. \tag{19}$$

And, the **set of all diagonal elements** is given as $\tilde{D} = \{\tilde{x} \in \tilde{U} : l_{\tilde{U}}(\tilde{x}) + r_{\tilde{U}}(\tilde{x}) = 1\}$.

3.2.1 Order Relations on $\tilde{U}$

According with (ATANASSOV; GARGOV, 1998) and (BUSTINCE; BURILLO; SORIA, 2003), this work considers the usual order relation $\leq_{\tilde{U}}$ given as:

$$(x_1, x_2) \leq_{\tilde{U}} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2, \tag{20}$$

for $\tilde{x}, \tilde{y} \in \tilde{U}$ such that $\tilde{0} = (0, 1) \leq_{\tilde{U}} \tilde{x}$ and $\tilde{1} = (1, 0) \geq_{\tilde{U}} \tilde{x}$.

This work studies the intuitionistic fuzzy index exploring related properties in the lattice $(\tilde{U}, \leq_{\tilde{U}}, \max_{\tilde{U}}, \min_{\tilde{U}}, \tilde{0}, \tilde{1})$ such that, for all $\tilde{x} \in \tilde{U}$, the following holds: $\max_{\tilde{U}}(x, y) = (\max(x, y), \min(x, y))$ and $\min_{\tilde{U}}(x, y) = (\min(x, y), \max(x, y))$.

Additionally, we also consider the preceq order expressed as follows:

$$(x_1, x_2) \preceq_{\tilde{U}} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \leq y_2. \tag{21}$$

3.2.2 Intuitionistic Fuzzy Index

The **intuitionistic fuzzy index** (IFIx) of an element $x \in \mathcal{X} \neq \emptyset$ related to an intuitionistic fuzzy set A, denoted by the following expression $\pi_A(x)$ is named as **hesitant degree** or **indeterminacy degree** of x in A. According with (XU; YAGER, 2009; ATANASSOV, 1999), for all $x \in \mathcal{X}$, the intuitionistic fuzzy index of x related to A is given by the following expression:

$$\pi_A(x) = 1 - \mu_A(x) - \nu_A(x), \tag{22}$$

for all $x \in \chi, \mu_A(x) + \nu_A(x) \leq 1.$ If $\pi_A(x) = 0$, A is a fuzzy set (SZMIDT; KACPRZYK, 2004).

Based on the above, the accuracy function $h_A : \chi \to U$ provides the accuracy degree of x in A, given as:

$$h_A(x) + \pi_A(x) = 1. \tag{23}$$

meaning that the largest $\pi_A(x)$ ($h_A(x)$), the higher the hesitancy (accuracy) degree of x in A.

Inherent properties of fuzzy implications related to IFIx are described in the following.

Definition 1. *(BUSTINCE; BARRENECHEA; MOHEDANO, 2004, Definition 3) An intuicionistic fuzzy implication $I_I : \tilde{U}^2 \to \tilde{U}$ is a function verifying, for all $(x,y),(x',y'),(z,t),(z',t') \in \tilde{U}$, the following properties:*
$\mathbf{I_{I_0}}$*: If $(x,y),(z,t) \in U$ are such that $x+y$ = 1 and $z+t$ = 1 then $\pi_{I_I((x,y),(z,t))}$ = 0;*
$\mathbf{I_{I_1}}$*: If $(x,y) \leq (x',y')$ then $I_I((x,y),(z,t)) \geq I_I((x',y'),(z,t))$;*
$\mathbf{I_{I_2}}$*: If $(z,t) \leq (z',t')$ then $I_I((x,y),(z,t)) \leq I_I((x,y),(z',t'))$;*
$\mathbf{I_{I_3}}$*: $I_I((0,1),(x,y)) = (1,0)$;*
$\mathbf{I_{I_4}}$*: $I_I((x,y),(1,0)) = (1,0)$;*
$\mathbf{I_{I_5}}$*: $I_I((1,0),(0,1)) = (0,1)$.*

Additionally, the group of properties of fuzzy implication which are truly intuitionistic are also considered in this work and reported in the following:
$\mathbf{I_{I_6}}$: $\pi_{I_I((x,y),(z,t))} \geq \max_{\tilde{U}}(1-x, 1-z)$;
$\mathbf{I_{I_7}}$: If $(x,y) = (z,t)$, then $\pi_{I_I((x,y),(z,t))} = \pi_{(x,y)}$;
$\mathbf{I_{I_8}}$: If $\pi_{(x,y)} = \pi_{(z,t)}$, then $\pi_{I_I((x,y),(z,t))} = \pi_{(x,y)}$.

In the following, aggregation operators are considered in order to define intuitinistic fuzzy implications also demanding idempotence and symmetry from boundary conditions and monotonicity properties.

Proposition 1. *(BUSTINCE; BARRENECHEA; MOHEDANO, 2004, Proposition 3) Let I be a fuzzy implication in J. Fodor sense and let I_N be the coimplication associated to I. Let M_1, M_2, M_3, M_4 be four idempotent aggregation operators satisfying the following conditions:*

$$M_1(x,y) + M_3(1-x, 1-y) \leq 1; \tag{24}$$
$$M_2(x,y) + M_4(1-x, 1-y) \geq 1, \forall x, y \in U. \tag{25}$$

Then $I_I : \tilde{U}^2 \to \tilde{U}$ given by

$$I_I((x,y),(z,t)) = (I(M_1(x,1-y), M_2(z,1-t)), I_N(M_3(y,1-x), M_4(t,1-z))) \tag{26}$$

is an Atanassov's intuitionistic fuzzy implication in sense of Fodor and Roubens (FODOR; ROUBENS, 1994).

The generalized intuitionist fuzzy index extending main properties in the especial group of fuzzy implications truly intuitionist, considered as follows:

Proposition 2. *(BUSTINCE; BARRENECHEA; MOHEDANO, 2004, Corollary 1(ii)) Let* $I_I : \tilde{U}^2 \to \tilde{U}$ *given by Eq.(26) according with conditions of Proposition 1. The function* I_I *verifies the following property*

$$\pi(I_I(\tilde{x},\tilde{y})) \quad \leq \quad \pi\left(I(N_S(y),z), I_N(N_S(x),t)\right). \tag{27}$$

Proposition 3. *(BUSTINCE; BARRENECHEA; MOHEDANO, 2004, Corollary 2) Let* $I_I : \tilde{U}^2 \to \tilde{U}$ *given by Eq.(26) according with conditions of Proposition 1. If* $I(x,y) \geq \min(x,y)$ *then* I_I *verifies property* $\mathbf{I_{I_6}}$*.*

Proposition 4. *Let* $I_I : \tilde{U}^2 \to \tilde{U}$ *given by Eq.(26) according with conditions of Proposition 1. The function* I_I *verifies property* $\mathbf{I_{I_7}}$ *and* $\mathbf{I_{I_8}}$*.*

Proof. I_I7: If $\tilde{x} = \tilde{y}$ then the following holds:

$$\pi(I_I(\tilde{x},\tilde{y})) =$$
$$N_S(I(N_S \circ I_N(M_3(N_S(x_1),x_2), M_4(N_S(x_1),x_2)), I(M_1(x_1,N_S(x_2)), M_2(x_1,N_S(x_2))))).$$

Moreover, since $M_1 = M_3 = \bigvee$ and $M_2 = M_4 = \bigwedge$ then we have that

$$\begin{aligned}
\pi(I_I(\tilde{x},\tilde{y})) &= N_S(I(N_S \circ I_N(N_S(x_1),x_2), I(x_1,N_S(x_2)))))\text{by } J_8\\
&= N_S(I(N_S(0), I(x_1,N_S(x_2)))\\
&= N_S(I(1, I(x_1,N_S(x_2)))\text{by } I_6\\
&= N_S(I(x_1,N_S(x_2))) = \pi(\tilde{x}).
\end{aligned}$$

I_I8: If $\pi(\tilde{x}) = \pi(\tilde{y})$ then $N_S(I(N_S(x_2),x_1)) = N_S(I(N_S(y_2),y_1))$. And, we have that:

$$\begin{aligned}
&\pi(I_I(\tilde{x},\tilde{y})) =\\
=\ & N_S(I(N_S \circ I_N(M_3(N_S(x_1),x_2), M_4(N_S(x_1),x_2)), I(M_1(x_1,N_S(x_2)), M_2(x_1,N_S(x_2)))))\\
=\ & N_S(I(N_S \circ I_N(N_S(x_1),x_2), I(x_1,N_S(x_2))))) \text{ by } J_8\\
=\ & N_S(I(N_S(0), I(x_1,N_S(x_2)))\\
=\ & N_S(I(1, I(x_1,N_S(x_2))) \text{ by } I_6\\
=\ & N_S(I(x_1,N_S(x_2)) = \pi(\tilde{x}).
\end{aligned}$$

Concluding, Proposition 4 is verified. □

3.2.3 Intuitionistic Conjugation Operators

The study of automorphisms is relevant since they can be used in the generation of new connectives, preserving main algebraic properties of classes of logical connectives (COSTA; BEDREGAL; NETO, 2011).

Definition 2. *(BUSTINCE; BURILLO; SORIA, 2003) The function $\Phi : \tilde{U} \rightarrow \tilde{U}$ is an* ***intuitionistic automorphism*** *in $\tilde{U}$ if it is bijective and, for all $\tilde{x}, \tilde{y}$, we have to $\tilde{x} \leq_{\tilde{U}} \tilde{y}$ if and only if $\Phi(\tilde{x}) \leq_{\tilde{U}} \Phi(\tilde{y})$.*

Just as $Aut(U)$ denotes the set and all the automorphisms in U, $Aut(\tilde{U})$ indicates the set and all the intuitionistic automorphisms in $\tilde{U}$.

The action of $\Phi \in Aut(\tilde{U})$ in a function $f_I : \tilde{U}^n \rightarrow \tilde{U}$ is a function $f_I^{\Phi} : \tilde{U} \rightarrow \tilde{U}$, called **intuitionistic conjugated of** f_I, defined for all $\tilde{x}_1, \ldots, \tilde{x}_n \in \tilde{U}$ for expression:

$$f_I^{\Phi}(\tilde{x}_1, \ldots, \tilde{x}_n) = \Phi^{-1}(f_I(\Phi(\tilde{x}_1), \ldots, \Phi(\tilde{x}_n))). \tag{28}$$

According with (COSTA; BEDREGAL; NETO, 2011, Theorem 17), let $\phi : U \rightarrow U$ be an automorphism on U. Then, for all $x \in U$, a **ϕ-representable automorphism** $\Phi : \tilde{U} \rightarrow \tilde{U}$ is defined by

$$\Phi(\tilde{x}) \;\; = \;\; (\phi(l_{\tilde{U}}(\tilde{x})), 1 - \phi(1 - r_{\tilde{U}}(\tilde{x}))). \tag{29}$$

Example 3. *Let $\phi : U \rightarrow U \in Aut(U)$ defined by $\phi_n(x) = x^n$ and let $\Phi_n : \tilde{U} \rightarrow \tilde{U}$ be a ϕ-representable automorphism given as $\Phi_n(x_1, x_2) = (x_1^n, 1 - (1 - x_2)^n)$. For instance, when $n = 2$ we have that $\Phi(\tilde{x}) = (x_1^2, 2x_2 + x_2^2)$ is a ϕ-automorphism obtained according with Eq.(29) in $Aut(\tilde{U})$.*

3.2.4 Intuitionistic Dual Operators

An **intuitionistic fuzzy negation** (IFN) $N_I : \tilde{U} \rightarrow \tilde{U}$ satisfies, for all $\tilde{x}, \tilde{y} \in \tilde{U}$, the following properties:

$\mathbf{N}_I$ **1:** $N_I(\tilde{0}) = N_I(0,1) = \tilde{1}$ and $N_I(\tilde{1}) = N_I(1,0) = \tilde{0}$;

$\mathbf{N}_I$ **2:** If $\tilde{x} \geq \tilde{y}$ then $N_I(\tilde{x}) \leq N_I(\tilde{y})$.

Moreover, N_I is a strong intuitionistic fuzzy negation (SIFN) verifying the condition $\mathbf{N}_I\mathbf{3}$: $N_I(N_I(\tilde{x})) = \tilde{x}$, $\forall \tilde{x} \in \tilde{U}$. Additionally, If N_I as IFN, the N_I-dual intuitionistic function $\tilde{f}_{N_I} : \tilde{U}^n \rightarrow \tilde{U}$ is given by:

$$\tilde{f}_{N_I}(\tilde{\mathbf{x}}) = N_I(\tilde{f}(N_I(\tilde{x}_1), \ldots, N_I(\tilde{x}_n))), \forall \tilde{\mathbf{x}} = (\tilde{x}_1, \ldots, \tilde{x}_n) \in \tilde{U}^n. \tag{30}$$

And, by (BACZYNSKI, 2004), taking a SFN $N : U \rightarrow U$, a IFN $N_I : \tilde{U} \rightarrow \tilde{U}$ such that

$$N_I(\tilde{x}) = (N(N_S(x_2)), N_S(N(x_1))), \tag{31}$$

is also SIFN called and N**-representable** IFN. Additionally, if $N = N_S$, Eq. (31) can be reduced to $N_I(\tilde{x}) = (x_2, x_1)$. Thus, we consider the complement of an IFS A given as

$$A' = \{(x, N(N_S(\nu_A(x)), N_S(N(\mu_A(x)))) : x \in \chi, \mu_A(x) + \nu_A(x)) \leq 1\} \subseteq \mathcal{A}_I.$$

4 GENERALIZED ATANASSOV'S INTUITIONISTIC FUZZY INDEX

In (BUSTINCE et al., 2011), the concept of the generalized Atanassov's intuitionistic fuzzy index (A-GIFIx) is characterized in terms of fuzzy implication operators. In addition, a constructive method with automorphisms is also proposed in (BARRENECHEA et al., 2009), together with some special properties of A-GIFIx.

In this chapter, we study properties of A-GIFIx, contribuiting with an incremental study of its duality and conjugation analysis.

4.1 Main Concepts

In (COSTA et al., 2016), main concepts of A-GIFIx are studied and its dual and conjugate constructions are also discussed.

Definition 3. *(BUSTINCE et al., 2011, Definition 1) A function* $\Pi : \tilde{U} \rightarrow U$ *is called a generalized intuitionistic fuzzy index associated with a SIFN* N_I $(A - GIFIx(N_I))$ *if, for all* $x_1, x_2, y_1, y_2 \in U$*, it holds that:*

$\mathbf{\Pi 1}$*:* $\Pi(x_1, x_2) = 1$ *if and only if* $x_1 = x_2 = 0$*;*

$\mathbf{\Pi 2}$*:* $\Pi(x_1, x_2) = 0$ *if and only if* $x_1 + x_2 = 1$*;*

$\mathbf{\Pi 3}$*: if* $(y_1, y_2) \preceq_{\tilde{U}} (x_1, x_2)$ *then* $\Pi(x_1, x_2) \leq \Pi(y_1, y_2)$

$\mathbf{\Pi 4}$*:* $\Pi(x_1, x_2) = \Pi(N_I(x_1, x_2))$ *when* N_I *is a SIFN.*

In particular, the following interpretations are held:

(i) Property $\Pi 1$ states the lack of information should be maximum, whenever there exists no information supporting/against a proposition;

(ii) In contrast, Property $\Pi 2$ states that when the membership and non-membership degrees are exactly complementary (related to fuzzy sets), the lack of information is minimum.

(iii) By $\Pi 3$, when the membership and the non-membership values increase, the lack of information decreases since the considered A-IFS is closer to being a FS.

(iv) And analyzing Property $\Pi 4$, no new information or knowledge is obtained only by negation. With respect to the use of the $\preceq_{\tilde{U}}$ in ordering instead of $\leq_{\tilde{U}}$, one can observe that the A-IFIx is neither increasing nor decreasing with respect to $\leq_{\tilde{U}}$, whereas it is decreasing with respect to $\preceq_{\tilde{U}}$.

A constructive method to obtain an A-GIFIx based on fuzzy (co)implications is proposed in (BUSTINCE et al., 2011) and reported below:

Proposition 5. *(BUSTINCE et al., 2011, Theorem 3) Let N_I be an N-representable IFN obtained by a SFN N. A function $\Pi : \tilde{U} \to U$ is a A-GIFIx(N_I) if and only if there exists a function $I : U^2 \to U$ verifying* $\mathbf{I1}$*,* $\mathbf{I8}$*,* $\mathbf{I9}$ *and* $\mathbf{I10}$ *such that*

$$\Pi_I(\tilde{x}) \quad = \quad N(I(N_S(x_2), x_1)), \forall \tilde{x} = (x_1, x_2) \in \tilde{U}. \tag{32}$$

The N_S-dual construction related to Proposition6 is considered in the following:

Proposition 6. *(COSTA et al., 2016, Proposition 1) Let N_I be an N-representable IFN obtained by a SFN N. A function $\Pi : \tilde{U} \to U$ is a A-GIFIx(N_I) if and only if there exists a function $J : U^2 \to U$ verifying* $\mathbf{J1}$*,* $\mathbf{J8}$*,* $\mathbf{J9}$ *and* $\mathbf{J10}$ *such that*

$$\Pi_J(\tilde{x}) \quad = \quad J(N(1 - x_2), N(x_1)), \forall \tilde{x} = (x_1, x_2) \in \tilde{U}. \tag{33}$$

Proof. (⇒) Let $J : U^2 \to U$ be a function verifying $\mathbf{J2}$, $\mathbf{J8}$, $\mathbf{J9}$ and $\mathbf{J10}$. It holds that:

$$\begin{aligned}
\mathbf{\Pi1} : \Pi_J(x_1,x_2) = 1 &\Leftrightarrow J(N(1-x_2), N(x_1)) = 1 \text{ (by Eq.(33))}\\
&\Leftrightarrow N(1-x_2) = 1 \, and \, N(x_1) = 0 \Leftrightarrow x_2 = 1 \, and \, x_1 = 1 \text{(by } \mathbf{J10}\text{)}\\
\mathbf{\Pi2} : \Pi_J(x_1,x_2) = 0 &\Leftrightarrow J(N(1-x_2), N(x_1)) = 0 \text{ (by Eq.(33))}\\
&\Leftrightarrow N(1-x_2) \geq N(x_1)\\
&\Leftrightarrow x_1 + x_2 \leq 1 \text{ and } x_1 + x_2 \geq 1 \Leftrightarrow x_1 + x_2 = 1 \text{ (by } \mathbf{J8} \text{ and Eq.(62))}\\
\mathbf{\Pi3} : (y_1,y_2) \preceq (x_1,x_2) &\Rightarrow y_1 \leq x_1 \, and \, y_2 \leq x_2 \text{ by Eq.(21)}\\
&\Rightarrow N(x_1) \geq N(y_1) \, and \, N(1-x_2) \leq N(1-y_2) \text{ by } \mathbf{N2}\\
&\Rightarrow J(N(1-x_2), N(x_1)) \leq J(N(1-y_2), N(y_1)) \text{ by } \mathbf{I1}\\
&\Rightarrow \Pi_J(x_1,x_2) \leq \Pi_J(y_1,y_2) \text{ by Eq.(33)}
\end{aligned}$$

When N_I is a SIFN,

$$\begin{aligned}
\mathbf{\Pi4} : \Pi_J(N_I(x_1,x_2)) &= \Pi_J(N(N_S(x_2)), N_S(N(x_1))) \text{ by Eq.(31)}\\
&= (J(x_1, 1-x_2)) \text{ by Eq.(33)}\\
&= (J(N(1-x_2)), N(x_1)) \text{ by } \mathbf{I9}\\
&= \Pi_J(x_1,x_2) \text{ by Eq.(33)}
\end{aligned}$$

(⇐) Consider the function $J(x_1,x_2) = \begin{cases} 1, & \text{if } x_1 > x_2,\\ \Pi_J(N(x_2), 1-N(x_1)), & \text{otherwise.} \end{cases}$

The following holds:

$$\mathbf{J2}: y_1 \geq y_2 \Leftrightarrow J(x,y_1) = \begin{cases} 1, & \text{if } x > y_1, \\ \Pi_J(N(y_1), 1-N(x)), & \text{otherwise; by Eq.(33)} \end{cases}$$
$$\geq \begin{cases} 1, & \text{if } x > y_2, \\ \Pi(N(y_2), 1-N(x)), & \text{otherwise; by } \mathbf{\Pi 3} \end{cases}$$
$$= J(x,y_2); \text{ by Eq.(33).}$$

$\mathbf{J8}$: Straightforward.

$$\mathbf{J9}: J(N(x_2), N(x_1)) = \begin{cases} 1, & \text{if } N(x_2) > N(x_1), \\ \Pi_J(x_1, 1-x_2), & \text{otherwise; by Eqs.(33) and (31)} \end{cases}$$
$$= \begin{cases} 1, & \text{if } x_1 \geq x_2, \\ \Pi_J(N_I(N(x_2), 1-N(x_1)))), & \text{otherwise, by } \mathbf{\Pi 4} \end{cases}$$
$$= \begin{cases} 1, & \text{if } x_1 \geq x_2, \\ \Pi_J(N(x_2), 1-N(x_1))), & \text{otherwise, by Eq.(33)} \end{cases}$$
$$= J(x_1,x_2), \text{whenever } N \text{ is a SFN.}$$

$$\mathbf{J10}: J(x_1,x_2) = 1 \Leftrightarrow \Pi_J(N(x_2), 1-N(x_1)) = 1 \text{by Eq.(33)}$$
$$\Leftrightarrow N(x_2) = 1-N(x_1) = 0 \Leftrightarrow x_1 = 0\, and\, x_2 = 1 \text{by } \mathbf{\Pi 1}.$$

Therefore, Proposition 6 holds. □

Theorem 3. *Based on conditions of Propositions 5 and 6 , when (I,J) is a pair of mutual N-dual function, meaning that $I_N = J$ or $J_N = I$, the following holds:*

$$\Pi_I(\tilde{x}) = \Pi_J(\tilde{x}), \ \forall \tilde{x} = (x_1,x_2) \in \tilde{U}. \tag{34}$$

Proof. By Proposition 5, we have that:

$$\Pi_I(\tilde{x}) = N(I(N_S(y),x)) = N(J_N(N_S(y),x)) = J(N(N_S(y),N(x)) = \Pi_J(\tilde{x})$$

Therefore, Theorem 3 is verified. □

4.1.1 Dual Operators

In this section we study the duality and conjugation properties related to A-GIFIx.

Theorem 4. *Let N be a SFN and N_I be its corresponding N-representable SIFN. For an A-GIFIx(N) $\Pi : \tilde{U} \to U$ the following holds:*

$$(\Pi)_N(\tilde{x}) = N(\Pi(\tilde{x})), \forall \tilde{x} \in \tilde{U}. \tag{35}$$

Proof. By Eq.(31) and Property $\mathbf{\Pi 4}$, $(\Pi)_N(\tilde{x}) = N(\Pi(N_I(\tilde{x}))) = N(\Pi(\tilde{x}))$. □

Results in Proposition 7 are related to (COSTA et al., 2016, Proposition 2).

Proposition 7. *Let N be a SFN and N_I be its corresponding N-representable SIFN. For a A-GIFIx(N) $\Pi_I(\Pi_J) : \tilde{U} \to U$ the following holds:*

$$(\Pi_I)_N(\tilde{x}) = I(N_S(x_2), x_1); \quad (36)$$

$$(\Pi_J)_N(\tilde{x}) = N(J(x_1, N_S(x_2))), \forall \tilde{x} = (x_1, x_2) \in \tilde{U}. \quad (37)$$

Proof. For all $\tilde{x} \in \tilde{U}$, $(\Pi_I)_N(\tilde{x}) = N(\Pi_I(N_I(\tilde{x}))) = N(\Pi(\tilde{x})) = I(N_S(x_2), x_1)$. Its dual construction can be proved analogously. □

In diagrams of Figures 1 and 2 the following denotation is considered:
(i) $C(I)$ and $C(J)$ denote the classes of fuzzy implications and coimplications verifying the conditons in Propositions 5 and 6;
(ii) $C(N)$ denotes the class of strong fuzzy negations on U;
(iii) $C(\Pi)$ provides denotation to the class of all A-IFIx.
These interrelations summarize the results stated in Propositions 5 and 6, Theorem 4 and Proposition7.

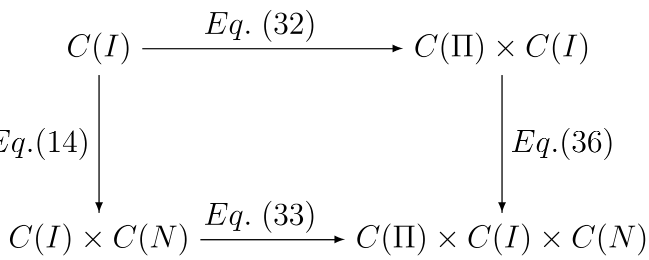

Figure 1 – A-GIFIx obtained by fuzzy implications and corresponding dual operator.

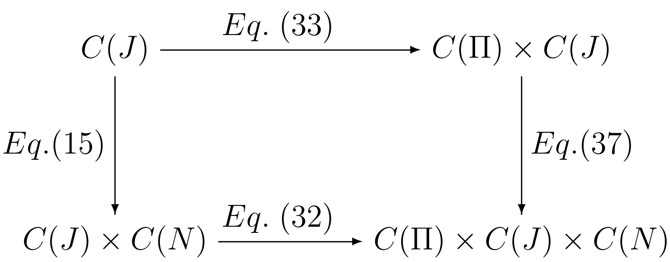

Figure 2 – A-GIFIx obtained by fuzzy coimplications and corresponding dual operator.

Consequently, one can describe hesitance and accuracy in terms of A-GIFIx. See, in (COSTA et al., 2016, Corollary 1), it is shown that the A-IFIx $\pi : \tilde{U} \to U$, can be defined as an $(A - GIFIx(N_{SI}))$ by considering the Lukaziewicz fuzzy implication $I_{LK} : U \to U$ given by the following expression

$$\Pi_{I_{LK}}(\tilde{x}) = \pi(\tilde{x}) = 1 - \mu_A(\mathrm{x}) - \nu_A(\mathrm{x})), \forall \mathrm{x} \in \chi; \quad (38)$$

and analogously, its N_S-dual construction can be given as follows:

$$(\Pi_{I_{LK}})_{N_S}(\tilde{x}) = h(\tilde{x}) = \mu_A(\mathrm{x}) + \nu_A(\mathrm{x})), \forall \mathrm{x} \in \chi. \quad (39)$$

Table 6 does not only illustrate Proposition 5, but also presents additional examples of A-GIFIx(N_{SI}) associated with the following fuzzy implications: Lukaziewicz, I_0, Reichenbach and Gaines-Rescher.

Table 6 – Generalized intuitionistic fuzzy index associated with the standard negation.

Pairs of N_S-Dual Fuzzy (Co)Implications	**N_{SI}-Dual A-GIFIx**
$I_{LK}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ 1-x+y, & \text{otherwise}; \end{cases}$ $J_{LK}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ y-x, & \text{otherwise}; \end{cases}$	$\Pi_{LK}(x,y)=1-x-y$ $(\Pi_{LK})_{N_{SI}}(x,y)=x+y$
$I_{KD}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ \max(1-x,y), & \text{otherwise}; \end{cases}$ $J_{KD}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ \min(1-x,y), & \text{otherwise}; \end{cases}$	$\Pi_{KD}(x,y)=1-\max(x,y)$ $(\Pi_{KD})_{N_{SI}}(x,y)=\max(x,y)$
$I_{RB}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ 1-x+xy, & \text{otherwise}; \end{cases}$ $J_{RB}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ 1-x-y+xy, & \text{otherwise}; \end{cases}$	$\Pi_{RB}(x,y)=1-x-y+xy$ $(\Pi_{RB})_{N_{SI}}(x,y)=y-xy$
$I_{GR}(x,y)=\begin{cases} 1, & \text{if } x \leq y, \\ 0, & \text{otherwise}; \end{cases}$ $J_{GR}(x,y)=\begin{cases} 0, & \text{if } x \geq y, \\ 1, & \text{otherwise}; \end{cases}$	$\Pi_{GR}(x,y)=1$ $(\Pi_{GR})_{N_{SI}}(x,y)=0$

4.1.2 Conjugate Operators

In the following, we study the action of automorphisms on A-GIFIx obtained by fuzzy (co)implications.

Proposition 8. *(COSTA et al., 2016, Prop. 3) Let $\Phi \in Aut(\tilde{U})$ be a ϕ-representable automorphism, $N^\phi : U \to U$ be the ϕ-conjugate of a SFN N. A function ${\Pi_G}^\Phi : \tilde{U} \to U$ is a $A-GIFIx(N_I^\phi)$ given by*

$$\Pi^\Phi(x_1, x_2) \;\; = \;\; (\phi^{-1}(\Pi(\phi(x_1)), 1-\phi(1-x_2)), \quad (40)$$

whenever $\Pi_G : \tilde{U} \to \tilde{U}$ is also an A-GIFIx(N_I).

Proof. Let $\Phi : \tilde{U} \to U$ be a representable ϕ-automorphism and $\Pi_G : \tilde{U} \to \tilde{U}$ be an

A-GIFIx(N_I). It holds that:

$$\begin{aligned}
\mathbf{\Pi1}: \Pi^{\Phi}_G(x_1,x_2)=1 &\Leftrightarrow \phi^{-1}(\Pi_G(\phi(x_1),1-\phi(1-x_2)))=1 \text{ (by Eq.(40))}\\
&\Leftrightarrow \Pi_G(\phi(x_1),1-\phi(1-x_2))=1 \text{ (by Eq.(28))}\\
&\Leftrightarrow \phi(x_1)=0\, and\, 1-\phi(1-x_2)=0 \text{ (by } \mathbf{\Pi1})\\
&\Leftrightarrow x_1=0\, and\, x_2=0 \text{ (by } \mathbf{A1})
\end{aligned}$$

$$\begin{aligned}
\mathbf{\Pi2}: \Pi^{\Phi}_G(x_1,x_2)=0 \quad &\Leftrightarrow \phi^{-1}(\Pi_G(\phi(x_1),1-\phi(1-x_2)))=0 \text{ (by Eq.(40))}\\
&\Leftrightarrow \Pi_G(\phi(x_1),1-\phi(1-x_2))=0 \text{ (by Eq.(28))}\\
&\Leftrightarrow \phi(x_1)+1-\phi(1-x_2)=1 \text{ (by } \mathbf{\Pi2})\\
&\Leftrightarrow \phi(x_1)=\phi(1-x_2) \Leftrightarrow x_1=1-x_2 \Leftrightarrow x_1+x_2=1\\
\mathbf{\Pi3}: (x_1,x_2)\preceq(y_1,y_2) &\Rightarrow x_1\le y_1\, and\, x_2\le y_2 \text{ by Eq.(21)}\\
&\Rightarrow \phi(x_1)\le\phi(y_1)\, and\; 1-\phi(1-x_2)\le 1-\phi(1-y_2) \text{ by } \mathbf{A1}\\
&\Rightarrow \Pi_G(\phi(x_1),1-\phi(1-x_2))\le \Pi_G(\phi(y_1),1-\phi(1-y_2)) \text{ by } \mathbf{\Pi3}\\
&\Rightarrow \phi^{-1}\left(\Pi_G(\phi(x_1),1-\phi(1-x_2))\right)\le \phi^{-1}\left(\Pi_G(\phi(y_1),1-\phi(1-y_2))\right)\\
&\Rightarrow \Pi^{\Phi}_G(x_1,x_2)\le \Pi^{\Phi}_G(y_1,y_2) \text{ by Eq.(40)}
\end{aligned}$$

Let N_I be a SIFN obtained by a SFN N, according with Eq.(31) and N_I^{Φ} be its $\Phi-$conjugate function. Therefore, it holds that:

$$\begin{aligned}
\mathbf{\Pi4}: \Pi^{\Phi}_G\left(N_I^{\Phi}(x_1,x_2)\right) &= \Pi_G{}^{\Phi}\left(N_I^{\Phi}(x_1,x_2)\right) \text{ (by Eq.(40))}\\
&= \phi^{-1}\left(\Pi_G(\Phi\circ\Phi^{-1}(N_I(\Phi(x_1,x_2))))\right) \text{ (by Eqs.(40) and (28))}\\
&= \phi^{-1}\left(\Pi_G(N_I(\Phi(x_1,x_2))))\right) \text{ (by } \mathbf{\Pi4})\\
&= \phi^{-1}\left(\Pi_G(\Phi(x_1,x_2))\right)=\Pi(x_1,x_2)
\end{aligned}$$

Concluding, Proposition 8 is verified. □

Proposition 9. *Let $\phi\in Aut(U)$ be an automorphism, $N^{\phi}: U\to U$ be a ϕ-conjugate of a SFN $N: U\to U$ and $I^{\phi}: U^2\to U$ be a ϕ-conjugate of $I: U^2\to U$. A function $\Pi_{I^{\phi}}(\Pi_{J^{\phi}}): \tilde{U}\to U$ given by*

$$\Pi_{I^{\phi}}(x_1,x_2) = N^{\phi}(I^{\phi}(1-x_2,x_1)), \tag{41}$$
$$\Pi_{J^{\phi}}(x_1,x_2) = J^{\phi}(N^{\phi}(1-x_2),N^{\phi}(x_1)), \tag{42}$$

is an A-GIFIx(N) whenever $\Pi_I(\Pi_J): \tilde{U}\to\tilde{U}$ is also an A-GIFIx(N).

Proof. For all $\tilde{x} = (x_1, x_2) \in \tilde{U}$, from Propositions 6 and 8 the results below are verified:

$$\begin{aligned}
\Pi_{I^\phi}(x_1,x_2) &= \phi^{-1}(\Pi_I(\phi(x_1), 1-\phi(1-x_2)) \text{ by Eq.(29)} \\
&= \phi^{-1}(N(I(\phi(1-x_2),\phi(x_1)))) \text{ by Eq.(32)} \\
&= \phi^{-1}(N(\phi\circ\phi^{-1})(I(\phi(1-x_2),\phi(x_1))) \\
&= N^\phi(I^\phi(1-x_2,x_1)) \text{ by Eq.(3)}
\end{aligned}$$

Analogously, its dual construction can be proved. Concluding, Proposition 9 holds. □

The main results in Propositions 4 and 5 together with Propositions 8 and 9 are summarized in the diagrams below (figures 3 and 4):

In diagrams of Figures 3 and 4 the following denotations are considered:
(i) $C(I)$ and $C(J)$ denote the classes of fuzzy implications and coimplications verifying the conditons in Propositions 32 and 33;
(ii) $Aut(U)$ denotes the class of automorfims on U;
(iii) $C(\Pi)$ provides denotation to the class of all A-GIFIx.
These interrelations summarize the results stated in Propositions 8 and 9.

$$\begin{array}{ccc}
\mathcal{C}(I) & \xrightarrow{Eq.\ (32)} & \mathcal{C}(\Pi)\times\mathcal{C}(I) \\
\Big\downarrow Eq.(3) & & \Big\downarrow Eqs.(29)(41) \\
\mathcal{C}(I)\times Aut(U) & \xrightarrow{Eq.\ (32)} & \mathcal{C}(\Pi)\times\mathcal{C}(I)\times Aut(\tilde{U})
\end{array}$$

Figure 3 – A-GIFIx obtained by fuzzy implications and conjugate operator.

$$\begin{array}{ccc}
\mathcal{C}(J) & \xrightarrow{Eq.\ (33)} & \mathcal{C}(\Pi)\times\mathcal{C}(J) \\
\Big\downarrow Eq.(3) & & \Big\downarrow Eqs.(29)(42) \\
\mathcal{C}(J)\times Aut(U) & \xrightarrow{Eq.\ (33)} & \mathcal{C}(\Pi)\times\mathcal{C}(J)\times Aut(\tilde{U})
\end{array}$$

Figure 4 – A-GIFIx obtained by fuzzy complications and conjugate operator.

See in Table 7, instances of A-GIFIx associated to ϕ-conjugate implications described in Table 6.

4.1.3 A-GIFIx (S,N)-implications and (T,N)-coimplications

In the following, the classes of (S,N)-implications and (T,N)-coimplications are considered in order to obtain new expressions of A-GIFIx.

Table 7 – A-GIFIx(N_{SI}) associated with the automorphisms $\phi(x) = x^2 \; and \; \phi^{-1} = \sqrt{x}$.

Fuzzy Implications	$A - GIFIx(N_{SI})$
$I^{\phi}_{KD}(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{\max(1-x^2, y^2)}, \text{otherwise}; \end{cases}$ $J^{\phi}_{KD}(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ \sqrt{\min((1-x)^2, y^2)}, \text{otherwise}; \end{cases}$	$\Pi_{I^{\phi}_{KD}}(x,y) = 1 - \sqrt{\max(x^2, 1-(1-y)^2)}$ $\Pi_{J^{\phi}_{KD}}(x,y) = \sqrt{1 - \max(x^2, 1-(1-y)^2))}$
$I^{\phi}_{LK}(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{1-x^2+y^2}, \text{ otherwise}; \end{cases}$ $J^{\phi}_{LK}(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ \sqrt{1-x^2+y^2}, \text{ otherwise}; \end{cases}$	$\Pi_{I^{\phi}_{LK}}(x,y) = 1 - \sqrt{1+x^2-(1-y)^2}$ $\Pi_{J^{\phi}_{LK}}(x,y) = \sqrt{x^2-(1-y)^2}$
$I^{\phi}_{RB}(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ \sqrt{1-x^2+x^2y^2}, \text{otherwise}; \end{cases}$ $J^{\phi}_{RB}(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ \sqrt{y^2-x^2y^2}, \text{otherwise}; \end{cases}$	$\Pi_{I^{\phi}_{RB}}(x,y) = 1 - \sqrt{1-(1-y)^2(1-x^2)}$ $\Pi_{J^{\phi}_{RB}}(x,y) = \sqrt{(1-y)^2(1-x^2)}$
$I^{\phi}_{GR}(x,y) = \begin{cases} 1, \text{ if } x \leq y, \\ 0, \text{otherwise}; \end{cases}$ $J^{\phi}_{GR}(x,y) = \begin{cases} 0, \text{ if } x \geq y, \\ 1, \text{otherwise}; \end{cases}$	$\Pi_{I^{\phi}_{GR}}(x,y) = 0$ $\Pi_{J^{\phi}_{GR}}(x,y) = 1$

Proposition 10. *(COSTA et al., 2016, Prop. 6) Let N be an SFN. A function $\Pi : \tilde{U} \to U$ is an A-GIFIx(N) if and only if there exists an S-implication (T-coimplication) $I_S(J_T)$: $U^2 \to U$ such that the following holds:*

$$\Pi_{I_{S,N}}(x_1, x_2) = S(N_S(x_2), N(x_1))); \quad (43)$$

$$\Pi_{J_{T,N}}(x_1, x_2) = T(N_S(x_2), N(x_1)). \quad (44)$$

Proof. Since N is a SFN, it holds that $(\Pi_{I_S})_N(x_1, x_2) = N(\Pi_{I_S}(N_I(x_1, x_2))) = N(\Pi_{I_S}(N \circ N_S(x_2), N_S \circ N(x_2))) = I_S(N(x_1), N \circ N_S(x_2)) = I_S(N_S(x_2), x_1) = \Pi_{I_S}(x_1, x_2)$. Moreover, $N(I_{S,N}(1 - x_2, x_1)) = S_N(N_S(x_2), N(x_1)))$. Therefore, $\Pi_{J_{T,N}}(x_1, x_2) = J_{T,N}(x_2, 1 - x_1)) = T(N_S(x_2), N(x_1))$, for all $(x_1, x_2) \in \tilde{U}$. □

In the following, by considering the standard negation, it is possible to obtain an A-GIFIx making use of t-(co)norms:

Corollary 1. *When $N = N_S$, Eq.(43) can be expressed as*

$$\Pi_{I_{S,N_S}}(x_1, x_2) = N_S(S(x_1, x_2)); \quad (45)$$

$$\Pi_{J_{T,N_S}}(x_1, x_2) = N_S(T_{N_S}(x_1, x_2)). \quad (46)$$

The functions in Table 7 area examples of (S, N)-implications and their dual constructions are (T, N)-coimplications.

4.2 Generation of Atanassov's Intuitionistic Fuzzy Entropy

In this section, the study of the Atanassov's intuitionistic fuzzy entropy follows from results stated in (BUSTINCE et al., 2011).

Definition 4. *(BUSTINCE et al., 2011, Definition 2) A real function $E : \mathcal{A}_I \to U$ is called an Atanassov's intuitionistic fuzzy entropy (A-IFE) if the following properties are verified:*

E1: $E(A) = 0$ *if and only if* $A \in \mathcal{A}$*,*

E2: $E(A) = 1$ *if and only if* $\mu_A(\mathrm{x}) = \nu_A(\mathrm{x}) = 0$*,* $\forall \mathrm{x} \in \chi$*,*

E3: $E(A) = E(A')$*,*

E4: *if* $A \preceq B$ *then* $E(A) \geq E(B), \forall A, B \in \mathcal{A}_I$*.*

According with (BUSTINCE et al., 2011), some interpretations of Definition 4:

(i) By **E1**, the lack of information should be the maximum whenever there is not information supporting a proposition, and if there is not information against the same proposition;

(ii) In opposite position, by **E2**, the lack of information is the minimum when we are dealing with a fuzzy set, meaning that the membership and non-membership degrees are exactly complementary;

(iii) As the third axim, **E3** states that the lack of information decreases if the membership and the non-membership values increase meaning that such A-IFS is closer to being a fuzzy set;

(iv) And, by last axiom, **E4** guarantees no new information can be obtained only by negation operator.

Proposition 11. *Let Φ be a ϕ-representable automorphism in $Aut(\tilde{U})$ and $E : \mathcal{A}_I \to U$ be an A-IFE. Then, for all $A \in \mathcal{A}$, the Φ-conjungate function $E^{\Phi} : \mathcal{A} \to U$ is also an A-IFE.*

Proof. Let A be an IFS. Taking $\Phi(A) = (\phi(\mu_A(\mathrm{x}), 1 - \phi(1 - \nu_A(\mathrm{x}))), \forall \mathrm{x} \in \chi$, the following is verified:

E1: If $E^{\Phi}(A) = 0$, $\phi^{-1}(E(\Phi(A))) = 0 \Leftrightarrow E(\Phi(A)) = 0 \Leftrightarrow \Phi(A) \in \mathcal{A} \Leftrightarrow A \in \mathcal{A}$.
E2: If $E^{\Phi}(A) = 1$, $\forall \mathrm{x} \in \tilde{U}$ the following holds

$$\begin{aligned} E^{\Phi}(A) = 1 &\Leftrightarrow \phi^{-1}(E(\phi(\mu_A(\mathrm{x})), 1 - \phi(1 - \nu_A(\mathrm{x}))) = 1 \\ &\Leftrightarrow E(\phi(\mu_A(\mathrm{x})), 1 - \phi(1 - \nu_A(\mathrm{x})) = 1, \\ &\Leftrightarrow \mu_A(\mathrm{x}) = 0 \text{ and } \phi(1 - \nu_A(\mathrm{x})) = 1 \\ &\Leftrightarrow \mu_A(\mathrm{x}) = 0 \text{ and } 1 - \nu_A(\mathrm{x}) = 1 \Leftrightarrow \mu_A(\mathrm{x}) = \nu_A(\mathrm{x}) = 0 \end{aligned}$$

E3: $E^{\Phi}(A') = \Phi^{-1}(E(\Phi(A'))) = \Phi^{-1}(E(\Phi(A')')) = \Phi^{-1}(E(\phi(A))) = E^{\Phi}(A)$.
E4: If $A \preceq B$, $\Phi(A) \preceq \Phi(B)$ then $E^{\Phi}(A) = \Phi^{-1}(E(\Phi(A)) \geq \Phi^{-1}(E(\Phi(B)) = E^{\Phi}(B)$. □

Properties related to A-IFE obtained by aggregation of A-GIFIx are discussed below by considering a finite set $\chi = \{x_1, \ldots, x_n\}$.

Proposition 12. *(BUSTINCE et al., 2011, Prop. 4) Let Ag be an aggregation on U, N be a SFN, Π be an A-GIFIx(N). Then, for all $A \in \mathcal{A}$, the mappings $E : \mathcal{A} \to U$ defines an Atanassov's intuitionistic fuzzy entropy (A-IFE) respectively expressed by*

$$E(A) \quad = \quad Ag_{i=1}^{n} \Pi(A(\mathrm{x}_i)) \tag{47}$$

Proposition 13. *Let Ag be an aggregation on U, N be a SFN, Π be an A-GIFIx(N) and $\phi \in Aut(U)$. Then, for all $A \in \mathcal{A}$, the mappings $E^{\Phi} : \mathcal{A} \to U$ expressed by*

$$E^{\Phi}(A) \quad = \quad Ag_{i=1}^{n} \Pi^{\Phi}(A(\mathrm{x}_i)), \forall \mathrm{x}_i \in \chi, \tag{48}$$

defines an Atanassov's intuitionistic fuzzy entropy.

Proof. Straightforward from Propositions 11 and 12. □

Let $C(E)$ be the class of all A-IFEs. The diagram below summarizes the main results related to the classes of A-GIFIx(N_I) and A-IFE.

The main results in Propositions 12 and 13 together with Propositions 8 and 9 are summarized in the diagram below (figure 5):

In diagram of Figure 5 the following denotation is considered:
(i) $C(\Pi)$ provides denotation to the class of all A-IFIx;
(ii) $Aut(U)$ denotes the class of all automorfims on U;
(iii) $C(E)$ provides denotation to the class of all entropy.
These interrelations summarize the results stated in Propositions 8 and 9.

In the following, an A-IFE is obtained from A-GIFIx as conceived in (BUSTINCE; BURILLO; SORIA, 2003), with respect to its dual and conjugate constructions.

Next two propositions report the main results from (COSTA et al., 2016) and (BUSTINCE et al., 2011).

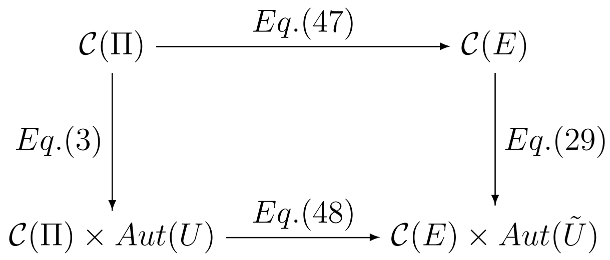

Figure 5 – Relatioship between $A - GIFIx(N)$ and $A - IFE\ Aut(\tilde{U})$

Proposition 14. *Consider $\phi \in Aut(U)$. Let $N : U \to U$ be a SFN, $Ag : U^n \to U$ be an aggregation function and $I_N : U^2 \to U$ be a N-dual operator of an implication $I : U^2 \to U$ which satisfies properties $\mathbf{I1}$, $\mathbf{I8}$, $\mathbf{I9}$ and $\mathbf{I10}$, as discussed in Proposition 5. Then, for all $A \in \mathcal{A}$, the mappings $E_I, E_{I^\Phi} : \mathcal{A} \to U$ defined by*

$$E_I(A) = Ag_{i=1}^{n} N(I(1 - \nu_A(\mathrm{x}_i), \mu_A(\mathrm{x}_i))), \quad (49)$$

$$E_{I^\phi}(A) = Ag_{i=1}^{n} N^\phi(I^\phi(1 - \nu_A(\mathrm{x}_i), \mu_A(\mathrm{x}_i))), \quad (50)$$

providing new expressions of A-IFEs obtained from an A-GIFIx(N_I).

Proof. Straightforward from Propositions 6 and 4, also taking Eq.(32) and (83). □

Proposition 15. *Consider $\phi \in Aut(U)$. Let $N : U \to U$ be a SFN, $Ag : U^n \to U$ be an aggregation function and $J_N : U^2 \to U$ be a N-dual operator of a coimplication $J : U^2 \to U$ satisfying properties $\mathbf{J2}$, $\mathbf{J8}$, $\mathbf{J9}$ and $\mathbf{J10}$, according with Proposition 6. Then, for all $A \in \mathcal{A}$, the mappings $E_J, E_{J^\phi} : \mathcal{A} \to U$ defined by*

$$E_J(A) = Ag_{i=1}^{n} J\left(N(1 - \nu_A(\mathrm{x}_i)), N(\mu_A(\mathrm{x}_i))\right), \forall x_i \in \chi, \quad (51)$$

$$E_{J^\phi}(A) = Ag_{i=1}^{n} J^\phi\left(N^\phi(1 - \nu_A(\mathrm{x}_i)), N^\phi(\mu_A(\mathrm{x}_i))\right), \forall x_i \in \chi, \quad (52)$$

are also Atanassov's intuitionistic fuzzy entropies (A-IFEs).

Proof. Straightforward from Propositions 14, 7 and 4, also taking Eq.(33). □

In diagram of Figure 6 the following denotation is considered:
(i) $C(I)$ denotes the class of all implications;
(ii) $C(\Pi)$ provides denotation to the class of all A-IFIx;
(iii) $Aut(U)$ denotes the class of all automorfims on U;
(iv) $C(E)$ provides denotation to the class of all entropy.
These interrelations summarize the results stated in Propositions 8 and 9.

Thus, the main results in Propositions 8 and 9 together with Propositions 12 and 13 are summarized in the diagram below (figure 6):

Proposition 16. *Let $E_J, E_{J_N} : \mathcal{A} \to U$ be A-IFEs according with Propositions 14*

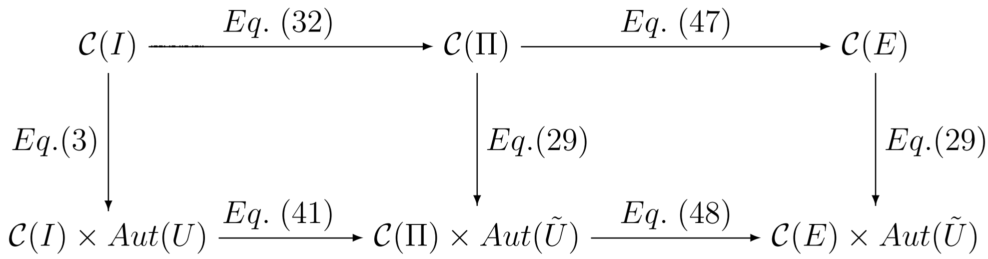

Figure 6 – Constructing A-IFE from classes of implications.

and 15. Then, for all $A \in \mathcal{A}$, the following holds:

$$E_{J_N}(A) = E_J(A) \text{ and } E_{I_N}(A) = E_I(A) \tag{53}$$

Proof. Straightforward from Proposition 6 and Eqs.(36) and (37) in Proposition 12. □

4.3 Case Study Expressing Fuzzy Entropy based on A-GIFIx

In order to illustrate and compare the above method ot obtain A-IFEs making use of aggregating A-GIFIxs, this section considers six expressions of A-IFEs introduced by M.Liu and H.Ren (LIU; REN, 2014). See algebraic expressions and related references reported in Table 8.

In this case study, E_7 is an IvFE obtained from Eq.(32) a by taking the arithmetic mean ($Ag = AM$) as the related aggregation operator. Thus the following holds:

$$E_7(A) = \frac{1}{n}\sum_{i=1}^{n} \Pi_I(\mathrm{x}_i).$$

Let n be a positive integer and $\chi = \{x_1, x_2, \ldots, x_n\}$ be a finite set in order to define the A-IFS A^n given by the following expression:

$$A^n = \{(\mathrm{x}_i, (\mu_A(\mathrm{x}_i)^n, 1-(1-\nu_A(\mathrm{x}_i)^n) : \mathrm{x}_i \in \chi\}.$$

Thus, when $\chi = \{6, 7, 8, 9, 10\}$ we obtain the following A-IFSs:

$$A = \{(6, 0.1, 0.8), (7, 0.3, 0.5), (8, 0.6, 0.2), (9, 0.9, 0.0), (10, 1.0, 0.0)\}$$

providing a characterization of linguistic variables treated as follows:

$A^{1/2}$ - "rather large" A - "quite large" A^2 - "large" A^3 - "very large" A^4 - "extremely large"

As a remark, we should mention that from axioms of the logical approach defining the A-GIFIx, the entropies of these IFSs follow the next pattern:

$$A^{1/2} < A < A^2 < A^3 < A^4 \ \Rightarrow\ E(A^{1/2}) > E(A) > E(A^2) > E(A^3) > E(A^4).$$

Table 8 – Interval-valued Fuzzy Entropies

Reference	Algebraic Expression
(YE, 2010)	$E_1(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\frac{1}{2}(2\mu_A(\mathrm{x}_i)+\Pi_A(\mathrm{x}_i))\cdot\log_2\frac{1}{2}(2\mu_A(\mathrm{x}_i)+\Pi_A(\mathrm{x}_i))+\frac{1}{2}(2\nu_A(\mathrm{x}_i)+\Pi_A(\mathrm{x}_i))\cdot\log_2\frac{1}{2}(2\nu_A(\mathrm{x}_i)+\Pi_A(\mathrm{x}_i))\right]$
(YE, 2010)	$E_2(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\left(\sqrt{2}cos(\mu_A(\mathrm{x}_i)-\nu_A(\mathrm{x}_i))\frac{\pi}{4}-1\right)\frac{1}{\sqrt{2}-1}\right]$
(VERMA; SHARMA, 2013)	$E_3(A) = \frac{1}{2n(\sqrt{e}-1)}\sum_{i=1}^{n}\left[2\mu_A(\mathrm{x}_i)+\Pi_A(\mathrm{x}_i))\cdot e^{1-\frac{1}{2}(2\mu_A(\mathrm{x}_i)+\Pi_A(\mathrm{x}_i))}+\frac{1}{2}(2\nu_A(\mathrm{x}_i)+\Pi_A(\mathrm{x}_i))e^{1-\frac{1}{2}(2\nu_A(\mathrm{x}_i)+\Pi_A(\mathrm{x}_i))}-1\right]$
(WEI; GAO; GUO, 2012)	$E_4(A) = \frac{1}{n}\sum_{i=1}^{n}cos\left(\frac{\mu_A(\mathrm{x}_i)-\nu_A(\mathrm{x}_i)}{(1+\Pi_A(x_i))}\frac{\pi}{4}\right)$
(YUE; JIA; YE, 2009)	$E_5(A) = \frac{1}{n}\sum_{i=1}^{n}cot\left(\frac{\pi}{4}+\frac{\lvert\mu_A(\mathrm{x}_i)-\nu_A(\mathrm{x}_i)\rvert}{(1+\Pi_A(\mathrm{x}_i))}\frac{\pi}{4}\right)$
(LIU; REN, 2014)	$E_6(A) = \frac{1}{n}\sum_{i=1}^{n}cot\left(\frac{\pi}{4}+(\lvert\mu_A(\mathrm{x}_i)-\nu_A(\mathrm{x}_i)\rvert*(1-\Pi_A(\mathrm{x}_i)))\frac{\pi}{4}\right)$

The expressions from E_1 to E_6 are considered in the application on the above IFSs from $A^{\frac{1}{2}}$ to A^4.

Each one of the A-GIFIxs is obtained by action of the arithmetic mean, taking into account the four A-GIFIxs Π_{LK}, Π_{RB}, Π_{GR} and Π_{KD} described in Table 6.

In addition, the entropy measures presented in Table 8 are compared based on these four expressions of the A-GIFIxs. They are also preserved by the proposed methodology related to the set of fuzzy implications $\{I_{GR}, I_{KD}, I_{RB}, I_{LK}\}$ as presented in Tables 9, 10, 11, 12, respectively.

Table 9 – A-IFE is obtained from the A-GIFIx ($\mathbf{\Pi_{GR}}$) with respect to N_S-dual construcion

$\mathbf{\Pi_{GR}}$	E_1	E_2	E_3	E_4	E_5	E_6	E_7
$A^{\frac{1}{2}}$	0,0146	0,0593	4,4133	0,9189	0,5322	0,8000	0,8000
A^1	0,0474	0,1403	4,2003	0,9184	0,5203	0,8000	0,8000
A^2	0,02178	0,2047	4,0138	0,9109	0,5045	0,8000	0,8000
A^3	0,0019	0,3595	3,9410	0,9061	0,4511	0,8000	0,8000
A^4	0,0176	0,4541	3,9004	0,9032	0,4271	0,8000	0,8000

We can see that the results of the entropy calculations follow the order of the implications proposed by Baczynski (BACZYNSKI; JAYARAM, 2007), such that $I_{LK} \leq I_{RB} \leq I_{KD}$. The I_{GR} was considered the largest of all implications, considering that it

Table 10 – A-IFE is obtained from the A-GIFIx ($\mathbf{\Pi_{KD}}$) with respect to N_S-dual construcion

$\mathbf{\Pi_{KD}}$	E_1	E_2	E_3	E_4	E_5	E_6	E_7
$A^{\frac{1}{2}}$	0,4690	0,4329	1,1199	0,8716	0,3954	0,4182	0,2352
A^1	0,4919	0,4854	1,0072	0,8747	0,3842	0,4088	0,24
A^2	0,4070	0,3942	0,8030	0,8472	0,3425	0,3552	0,224
A^3	0,3687	0,3571	0,5444	0,8287	0,2616	0,2829	0,1832
A^4	0,3383	0,3179	0,4138	0,8167	0,2221	0,2454	0,1635

Table 11 – A-IFE is obtained from the A-GIFIx ($\mathbf{\Pi_{RB}}$) with respect to N_S-dual construcion

$\mathbf{\Pi_{RB}}$	E_1	E_2	E_3	E_4	E_5	E_6	E_7
$A^{\frac{1}{2}}$	0,5095	0,4669	0,8474	0,8701	0,3847	0,3982	0,1757
A^1	0,5231	0,5003	0,7844	0,8732	0,3749	0,3911	0,1900
A^2	0,4427	0,4123	0,5697	0,8466	0,3407	0,3501	0,1733
A^3	0,3821	0,3508	0,4458	0,8281	0,2577	0,2748	0,1604
A^4	0,3430	0,3098	0,3697	0,8161	0,2193	0,2399	0,1528

assumes the highest and lowest values (0 or 1).
The implication I_{LK} was not the best indicated for this application, since its results do not follow the proposed order $E(A^{1/2}) > E(A) > E(A^2) > E(A^3) > E(A^4)$. We also point out that the proposed order has no relation to the axiom E4, Definition 4.

Table 12 – A-IFE is obtained from the A-GIFIx ($\mathbf{\Pi_{LK}}$) with respect to N_S-dual construcion

$\mathbf{\Pi_{LK}}$	E_1	E_2	E_3	E_4	E_5	E_6	E_7
$A^{\frac{1}{2}}$	0,5462	0,5051	0,5105	0,8659	0,3645	0,3685	0,0923
A^1	0,5496	0,4939	0,5054	0,8685	0,3564	0,3632	0,1200
A^2	0,4588	0,3952	0,4064	0,8436	0,3338	0,3407	0,132
A^3	0,3895	0,3330	0,3437	0,8262	0,2512	0,2643	0,1344
A^4	0,3466	0,2937	0,3044	0,8146	0,2141	0,2313	0,1359

5 INTERVAL-VALUED INTUITIONISTIC FUZZY LOGIC

This chapter provides a brief account on the Interval-valed Intuitionistic Fuzzy Logic (A-IvIFL) as proposed by Atanassov, keeping this work self-contained. Firstly, by reporting basic concepts of interval-valued automorphisms, interval-valued fuzzy negations on $\mathbf{U}$ and main properties of interval-valued fuzzy implications.

5.1 Basic Concepts of the Interval-Valued Fuzzy Logic

Based on interpretations provided by the interval-valued fuzzy set theory, the membership degree of an element $x \in \chi$ to a fuzzy set corresponds to a value in the considered membership interval. So, we cannot say in a precise way what that value is, meaning that we just provide bounds for it represented by the interval-valued membership function.

Let $\mathbb{U} = \{[x_1, x_2] \colon x_1, x_2 \in U$ and $x_1 \leq x_2\}$ be the set of all subintervals of the unit interval U. The projections $l_{\mathbb{U}}, r_{\mathbb{U}} : \mathbb{U} \to U$ are defined by

$$l_{\mathbb{U}}([x_1, x_2]) = x_1 \text{ and } r_{\mathbb{U}}, ([x_1, x_2]) = x_2, \ \forall x, y \in \mathbb{U} \tag{54}$$

and for $X \in \mathbb{U}$, $l_{\mathbb{U}}(X)$ and $r_{\mathbb{U}}(X)$ are also denoted by $\underline{X}$ and $\overline{X}$, respectively.

For each $x \in U$, the degenerate interval $[x, x]$ will be denoted by $\mathbf{x}$ and related set $\bar{D} = \{\mathbf{x} \colon x \in U\}$ denotes the set of all degenerate intervals on $\mathbb{U}$.

An interval-valued fuzzy set can be expressed as follows:

$$\mathbb{A} = \{(x, \mu_{\mathbb{A}}(x)) \colon x \in \chi \text{ and } \mu_{\mathbb{A}}(x) \in \mathbb{U}\}.$$

5.1.1 Order Relations on $\mathbb{U}$

Among different order relations to compare elements in IvFSs (GEHRKE; WALKER; WALKER, 1996), we take the component-wise **Kulisch-Miranker order** (or **product order**), given by:

$$X \leq_{\mathbb{U}} Y \Leftrightarrow \underline{X} \leq \underline{Y} \text{ and } \overline{X} \leq \overline{Y}, \ \forall X, Y \in \mathbb{U}.$$

Thus, $\mathbf{0} \leq_{\mathbb{U}} X \leq_{\mathbb{U}} \mathbf{1}$, for all $X \in \mathbb{U}$. Morover, for all $X, Y \in U$, by taking

$$\wedge(X, Y) = \{\max(x, y) \colon x \in X, y \in Y\} \text{ and } \vee(X, Y) = \{\min(x, y) \colon x \in X, y \in Y\}$$

the structured set $\mathbb{U} \equiv (\mathbb{U}, \leq_{\mathbb{U}}, \wedge, \vee, \mathbf{1}, \mathbf{0})$ is a lattice.

We also consider the relation $\preceq_{\mathbb{U}} \subseteq \mathbb{U} \times \mathbb{U}$ given as

$$X \preceq_{\mathbb{U}} Y \Leftrightarrow \overline{X} \leq \underline{Y}, \ \ \forall X, Y \in \mathbb{U}.$$

Therefore, we have that $X \preceq_{\mathbb{U}} Y \Rightarrow X \leq_{\mathbb{U}} Y$, $\forall X, Y \in \mathbb{U}$.

Since each interval $[\underline{X}, \overline{X}] \subseteq \mathbb{U}$ can be assigned uniquely to a point $(\underline{X}, \overline{X}) \in U \times U = U^2$, intervals can be ordered by means of pointwise orders in $U \times U$ induced by the partial order of intervals $\leq_{\mathbb{U}}$.

Remark that, when $K([0,1]) = \{(x, y) \in [0,1]^2 | x \leq y\}$, there is a natural bijection from $\mathbb{U}$ onto $K([0,1])$ resulting in the following

$$[\underline{X}, \overline{X}] \leq_{\mathbb{U}} [\underline{Y}, \overline{Y}] \Leftrightarrow (\underline{X}, \overline{X}) \leq_{U \times U} (\underline{Y}, \overline{Y}),$$

and meaning that a partial (linear) order on $\mathbb{U}$ induces a partial (linear) order on the other, $K([0,1])$.

However, a linear order of intervals is required in order to compare anyone of its elements on $\mathbb{U}$. Thus, we consider an order relation extending the partial order $\leq_{\mathbb{U}}$ to a linear order by applying the notion of an admissible order.

Definition 5. *(BUSTINCE et al., 2013) Let $(\mathbb{U}, \sqsubseteq)$ be a poset. The order $\sqsubseteq$ is called an admissible order, if*

(i) *$\sqsubseteq$ is a linear order on $\mathbb{U}$,*

(ii) *for all $X, Y \in \mathbb{U}$, $X \sqsubseteq Y$ whenever $X \leq_{\mathbb{U}} Y$.*

By Definition 5, an order $\sqsubseteq_{\mathbb{U}}$ is admissible, if it is linear and improve the order $\leq_{\mathbb{U}}$.

Example 4. *By considering the lexicographical order on $U \times U$, the order relations on $\sqsubseteq_{Lex1}, \sqsubseteq_{Lex2}$ on $\mathbb{U} \times \mathbb{U}$ given as :*

(i) $[a, b] \sqsubseteq_{Lex1} [c, d] \Leftrightarrow a < c \vee a = c \wedge b \leq d$*; and*

(ii) $[a, b] \sqsubseteq_{Lex2} [c, d] \Leftrightarrow b < d \vee b = d \wedge a \leq c$*,*

are admissible linear orders.

5.1.2 Conjugation Operators on $\mathbb{U}$

In this session, some concepts of interval automorphisms are presented. This study is the basis for obtaining the conjugated functions, used in this work.

An interval function $\mathbb{U} \rightarrow \mathbb{U}$ is an **interval automorphism** (IvA) if it is bijective and monotonic with respect to the product order, that is, $X \leq_{\mathbb{U}} Y$ if and only if $X \leq_{\mathbb{U}} Y$.

Let $Aut(\mathbb{U})$ be the set of all interval in $\mathbb{U}$. Interval automorphisms are closed for composition, that is, for all $(\in Aut(\mathbb{U}), \circ) \in Aut(\mathbb{U})$; and for all $\in Aut(\mathbb{U})$ there is the reverse automorphism $^{-1} \in \mathbb{U}$, such that $\circ^{-1} = Id_{\mathbb{U}}$. Thus, $(Aut(\mathbb{U}), \circ)$ is a group.

The action of an IvA $: \mathbb{U} \rightarrow \mathbb{U}$ about an interval function $F : \mathbb{U}^n \rightarrow \mathbb{U}$ is an interval function $F : \mathbb{U} \rightarrow \mathbb{U}$, called **interval conjugated** of F, defined by the expression:

$$F(X_1, \ldots, X_n) = \Phi^{-1}(F(\Phi(X_1), \ldots, \Phi(X_n))). \tag{55}$$

5.1.3 Dual Operators on $\mathbb{U}$

Interval-valued fuzzy neagtions and dual operators are considered in the following.

Definition 6. *(REISER et al., 2007) A interval function $\mathbb{N} : \mathbb{U} \rightarrow \mathbb{U}$ is an interval-valued fuzzy negation (IvFN) if, for all $X, Y \in \mathbb{U}$, it verifies the conditions:*

$\mathbb{N}1$**:** $\mathbb{N}([0,0]) = \mathbf{1}$*; and* $\mathbb{N}([1,1]) = \mathbf{0}$*;*

$\mathbb{N}2a$ *: If $X \geq Y$ then $\mathbb{N}(X) \leq \mathbb{N}(Y)$.*

$\mathbb{N}2b$ *: If $X \subseteq Y$ then $\mathbb{N}(X) \supseteq \mathbb{N}(Y)$.*

If $\mathbb{N}$ also satisfies the involutive property:

$\mathbb{N}3$ *: $\mathbb{N}(\mathbb{N}(X)) = X$, for all $X \in \mathbb{U}$,*

then $\mathbb{N}$ is called strong IvFN (REISER et al., 2007).

Definition 7. *(REISER; BEDREGAL; REIS, 2012) Let $\mathbb{N}$ be an interval fuzzy strong negation in $\mathbb{U}$ and $F : \mathbb{U}^n \leftrightarrow \mathbb{U}$ be an interval function. The **interval function $\mathbb{N}$-dual** of F is given by:*

$$F_{\mathbb{N}}(X_1, \ldots, X_n) = \mathbb{N}(F(\mathbb{N}(X_1), \ldots, \mathbb{N}(X_n))). \tag{56}$$

Example 5. *The interval extension of the standard fuzzy negation $\mathbb{N}_S : \mathbb{U} \rightarrow \mathbb{U}$ is given as:*

$$\mathbb{N}_S(X) = \mathbf{1} - X = [1 - \overline{X}, 1 - \underline{X}]. \tag{57}$$

5.1.4 Aggregation Operators on $\mathbb{U}$

An interval-valued extension of an aggregation function $\mathbb{A} : \mathbb{U}^n \rightarrow \mathbb{U}$ demands the following conditions:

$\mathbb{A}1$**:** $\mathbb{A}(\mathbf{X}) = \mathbf{0} \Leftrightarrow \mathbf{X} = (\mathbf{0}, \ldots, \mathbf{0})$; $\quad \mathbb{A}(\mathbf{X}) = \mathbf{1} \Leftrightarrow \mathbf{X} = (\mathbf{1}, \ldots, \mathbf{1})$;

$\mathbb{A}2$**:** If $\mathbf{X} = (X_1, \ldots, X_n) \leq_{\mathbb{U}^n} \mathbf{Y} = (Y_1, \ldots, Y_n)$ then $\mathbb{A}(\mathbf{X}) \leq_{\mathbb{U}} \mathbb{M}(\mathbf{Y})$;

$\mathbb{A}3$**:** $\mathbb{A}(\mathbf{X}_\sigma) = \mathbb{A}(X_{\sigma_1}, \ldots, X_{\sigma_n}) = \mathbb{A}(X_1, \ldots, X_n) = \mathbb{A}(\mathbf{X})$.

Interval-valued aggregations are idempotent if they also verify the following condition:

$\mathbb{A}4$: $\mathbb{A}(X, X) = X$, $\forall X \in \mathbb{U}$ (idempotency property).

Example 6. *Let* $\mathbb{A} : \mathbb{U}^n \to \mathbb{U}$ *be an idempotent IvA, together with the functions* $\bigwedge, \bigvee : \mathbb{U}^2 \to \mathbb{U}$*, given by:*

$$\bigwedge(X, Y) = [\wedge(\underline{X}, \underline{Y}), \wedge(\overline{X}, \overline{Y})] \quad \textit{and} \quad \bigvee(X, Y) = [\vee(\underline{X}, \underline{Y}), \vee(\overline{X}, \overline{Y})]. \tag{58}$$

The following inequation is held:

$$\bigwedge(X, Y) \leq A(X, Y) \leq \bigvee(X, Y), \; \forall X, Y \in \mathbb{U}. \tag{59}$$

In the following, the definition of an interval extension of conjuntive and disjuntive connectives on $\mathbb{U}$ is considered.

Definition 8. *(BEDREGAL et al., 2007) A function* $\mathbb{T}(\mathbb{S}) : \mathbb{U}^2 \to \mathbb{U}$ *is an Interval-valued T-norma (Interval-valued T-(co)norma) IvT (IvS) if, for all* $X, Y, Z \in \mathbb{U}$ *the following properties are verified:*

$\mathbb{T}1$*:* $\mathbb{T}(X, Y) = \mathbb{T}(Y, X)$*;*	$\mathbb{S}1$*:* $\mathbb{S}(X, Y) = \mathbb{S}(Y, X)$*;*
$\mathbb{T}2$*:* $\mathbb{T}(X(\mathbb{T}(Y, Z)) = \mathbb{T}(\mathbb{T}(X, Y), Z)$*;*	$\mathbb{S}2$*:* $\mathbb{S}(X(\mathbb{S}(Y, Z)) = \mathbb{S}(\mathbb{S}(X, Y), Z)$*;*
$\mathbb{T}3$*:* $\mathbb{T}(X, 1) = X$*;*	$\mathbb{S}3$*:* $\mathbb{S}(X, 0) = X$*;*
$\mathbb{T}4$*:* $\mathbb{T}(X, Y) \leq \mathbb{T}(X, Z)$ *if* $Y \leq Z$	$\mathbb{S}4$*:* $\mathbb{S}(X, Y) \leq (X, Z)$ *if* $Y \leq Z$*.*

Proposition 17. *(BEDREGAL; TAKAHASHI, 2006a) A function* $\mathbb{T}(\mathbb{S}) : \mathbb{U}^2 \to \mathbb{U}$ *is an Interval-valued T-norma (Interval-valued T-(co)norma) IvT (IvS) if there are* T_1*,* $T_2(S_1, S_2) : U^2 \to U$ *such that* $T_1(x, y) \leq T_2(x, y)$ *(*$S_1(x, y) \leq S_2(x, y)$*) and the following holds:*

$$\mathbb{T}(X, Y) = [T_1(\underline{X}, \underline{Y}), T_2(\overline{X}, \overline{Y})] \qquad \mathbb{S}(X, Y) = [S_1(\underline{X}, \underline{Y}), S_2(\overline{X}, \overline{Y})] \tag{60}$$

In Proposition 17, an interval-valued t-(co)norm can be considered as an interval representation of a t-(co)norm. This generalization fits with the fuzzy principle, meaning that the interval-valued membership degree can be thought of as an approximation of the degree of exact relevance related to a specialist.

Thus, an IvT $\mathbb{T}$ is t-representable by t-norms $\mathbb{T}1$ and $\mathbb{T}2$, in the sense as proposed in (DESCHRIJVER; KERRE, 2005; CORNELIS; DESCHRIJVER; KERRE, 2004). It is analogous stated in the dual construction of an IvS, as can be seen in (BEDREGAL; TAKAHASHI, 2006b).

5.1.5 Interval-valued Fuzzy (Co)Implications

Fuzzy (co)implications can then be naturally extended to an interval-based approach. In the following, we study the definition and the main properties of interval-valued fuzzy (co)implication.

Definition 9. *(BEDREGAL et al., 2007) A function* $\mathbb{I}(\mathbb{J}) : \mathbb{U}^2 \rightarrow \mathbb{U}$ *is a interval-valued fuzzy (co)implication if it satisfies the following conditions:*

$\mathbb{I}1$*: If* $X \leq Z$ *then* $\mathbb{I}(X,Y) \geq \mathbb{I}(Z,Y)$*;*	$\mathbb{J}1$*: If* $X \leq Z$ *then* $\mathbb{J}(X,Y) \geq \mathbb{J}(Z,Y)$*;*
$\mathbb{I}2$*: If* $Y \leq Z$ *then* $\mathbb{I}(X,Y) \leq \mathbb{I}(X,Z)$*;*	$\mathbb{J}2$*: If* $Y \leq Z$ *then* $\mathbb{J}(X,Y) \leq \mathbb{J}(X,Z)$*;*
$\mathbb{I}3$*:* $\mathbb{I}(\mathbf{0},Y) = \mathbf{1}$*;*	$\mathbb{J}3$*:* $\mathbb{J}(\mathbf{1},Y) = \mathbf{0}$*;*
$\mathbb{I}4$*:* $\mathbb{I}(X,\mathbf{1}) = \mathbf{1}$*;*	$\mathbb{J}4$*:* $\mathbb{J}(X,\mathbf{0}) = \mathbf{0}$*;*
$\mathbb{I}5$*:* $\mathbb{I}(\mathbf{1},\mathbf{0}) = \mathbf{0}$*;*	$\mathbb{J}5$*:* $\mathbb{J}(\mathbf{0},\mathbf{1}) = \mathbf{1}$*.*

Since real numbers may be identified with degenerate intervals in the context of interval mathematics, the boundary conditions that must be satisfied by the classical fuzzy implications can be naturally extended to interval fuzzy degrees, whenever degenerate intervals are considered. So, an interval-valued fuzzy (co)implicator $\mathbb{I}(\mathbb{J}) : \mathbb{U}^2 \rightarrow \mathbb{U}$ verifies $\mathbb{I}5$ ($\mathbb{J}5$) together with the following boundary conditions:

$\mathbb{I}0$: $\mathbb{I}(\mathbf{1},\mathbf{1}) = \mathbb{I}(\mathbf{0},\mathbf{0}) = \mathbb{I}(\mathbf{0},\mathbf{1}) = \mathbf{1}$; $\mathbb{J}0$: $\mathbb{J}(\mathbf{1},\mathbf{1}) = \mathbb{J}(\mathbf{1},\mathbf{0}) = \mathbb{J}(\mathbf{0},\mathbf{0}) = \mathbf{0}$;

Several reasonable properties may be required for fuzzy (co)implications. In this work, we consider the folowing ones:

$\mathbb{I}6$: $\mathbb{I}(\mathbf{1},Y) = Y$;	$\mathbb{J}6$: $\mathbb{J}(\mathbf{0},Y) = Y$.
$\mathbb{I}7$: $\mathbb{I}(X,\mathbb{I}(Y,Z))=\mathbb{I}(Y,I(X,Z))$;	$\mathbb{J}7$: $\mathbb{J}(X,\mathbb{J}(Y,Z))=\mathbb{J}(Y,J(X,Z))$;
$\mathbb{I}8$: $\mathbb{I}(X,Y)=\mathbf{1} \Leftrightarrow X \leq_{\mathbb{U}} Y$;	$\mathbb{J}8$: $\mathbb{J}(X,Y)=\mathbf{0} \Leftrightarrow X \geq_{\mathbb{U}} Y$;
$\mathbb{I}9$: $\mathbb{I}(X,Y) = \mathbb{I}(\mathbb{N}(Y),\mathbb{N}(X))$, $\mathbb{N}$ is a SIFN;	$\mathbb{J}9$: $\mathbb{J}(x,y) = \mathbb{J}(\mathbb{N}(Y),\mathbb{N}(X))$, $\mathbb{N}$ is a SIFN;
$\mathbb{I}10$: $\mathbb{I}(X,Y) = \mathbf{0} \Leftrightarrow X = \mathbf{1}$ and $Y = \mathbf{0}$;	$\mathbb{J}10$: $\mathbb{J}(X,Y) = \mathbf{1} \Leftrightarrow X = \mathbf{0}$ and $Y = \mathbf{1}$.

The conditions under which an interval-valued fuzzy (co)implication can be obtained by a fuzzy (co)implication is studied in the proposition below:

Proposition 18. *(BACZYNSKI; JAYARAM, 2007, Prop 21) A fuzzy (co)implication* $I(J) : U^2 \rightarrow U$ *satisfies properties* $\mathbb{I}1$ *(*$\mathbb{J}1$*) and* $\mathbb{I}2$ *(*$\mathbb{J}2$*) if and only if the interval fuzzy (co)implication* $\mathbb{I}$ *is given as*

$$\mathbb{I}(X,Y) = [I(\overline{X},\underline{Y}), I(\underline{X},\overline{Y})]; \qquad \mathbb{J}(X,Y) = [J(\overline{X},\underline{Y}), J(\underline{X},\overline{Y})]. \tag{61}$$

See Table 13, the interval-approach extension related to fuzzy (co)implications in Table 6 is presented. In addition, since the conditions of Proposition 18 are verified, these interval-valued fuzzy implications can be expressed by the corresponding fuzzy implications, as detailed in the following example.

Table 13 – Interval- valued Fuzzy Implications and $\mathbb{N}_S$-dual constructions.

Interval-valued Fuzzy Implications	**Interval-valued Fuzzy Coimplications**
$\mathbb{I}_{LK}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X \leq Y,\\ \mathbf{1}-X+Y, & \text{otherwise};\end{cases}$	$\mathbb{J}_{LK}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X \geq Y,\\ Y-X, & \text{otherwise};\end{cases}$
$\mathbb{I}_{KD}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X \leq Y,\\ \max(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$	$\mathbb{J}_{KD}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X \geq Y,\\ \min(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$
$\mathbb{I}_{RB}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X \leq Y,\\ \mathbf{1}-X+XY, & \text{otherwise};\end{cases}$	$\mathbb{J}_{RB}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X \geq Y,\\ Y-XY, & \text{otherwise};\end{cases}$
$\mathbb{I}_{GR}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X \leq Y,\\ \mathbf{0}, & \text{otherwise};\end{cases}$	$\mathbb{J}_{GR}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X \geq Y,\\ \mathbf{1}, & \text{otherwise};\end{cases}$
$\mathbb{I}_{30}(X,Y)=\begin{cases}\max(\mathbf{1}-X,Y,\mathbf{0.5}), & \text{if } \mathbf{0}<Y<X<\mathbf{1},\\ \max(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$	$\mathbb{J}_{30}(X,Y)=\begin{cases}\min(\mathbf{1}-X,Y,\mathbf{0.5}), & \text{if } \mathbf{0}<X<Y<\mathbf{1},\\ \min(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$

Example 7. *When Eq. (61) is applied to the extension of Reichbach fuzzy implications $\mathbb{I}_{RB}$ and its $\mathbb{N}_S$-dual construction $\mathbb{J}_{RB}$, we obtain the next expressions:*

$$\begin{aligned}
\mathbb{I}_{RB}(X,Y) &= \begin{cases}\mathbf{1}, & \textit{if } X \leq Y,\\ \mathbf{1}-X+XY, & \textit{otherwise};\end{cases} = \begin{cases}\mathbf{1}, & \textit{if } X \leq Y,\\ \mathbb{N}_S(X)+X\cdot Y, & \textit{otherwise};\end{cases}\\
&= \begin{cases}\mathbf{1}, & \textit{if } X \leq Y,\\ \left[N_S(\overline{X})+\overline{X}\cdot\underline{Y}, N_S(\underline{X})+\underline{X}\cdot\overline{Y}\right], & \textit{otherwise};\end{cases}\\
\mathbb{J}_{RB}(X,Y) &= \begin{cases}\mathbf{0}, & \textit{if } X \geq Y,\\ Y-XY, & \textit{otherwise};\end{cases} = \begin{cases}\mathbf{0}, & \textit{if } X \geq Y,\\ \mathbb{N}_S(X)\cdot Y, & \textit{otherwise};\end{cases}\\
&= \begin{cases}\mathbf{0}, & \textit{if } X \geq Y,\\ \left[N_S(\underline{X})\cdot\overline{Y}, N_S(\overline{X})\cdot\underline{Y}\right], & \textit{otherwise};\end{cases}
\end{aligned}$$

According with the conditions of Proposition 18, analogous interval representations can be obtained to other (co)implications presented in Table 13.

5.2 Main Concepts of the Interval-Valued Intuitionistic Fuzzy Logic

Since Atanassov introduced the interval-valued fuzzy set theory, fruitful results have been achieved, introducing several basic operations, expanding both depth and scope, effectively aggregation and fuzzy connectives.

5.2.1 Interval-valued Intuitionistic Fuzzy Sets

Based on (ATANASSOV; GARGOV, 1998) and later in (CORNELIS; DESCHRIJVER; KERRE, 2004), we briefly report main concepts and properties on interval-valued Atanassov's intuitionistic fuzzy sets (A-IvIFSs shortly).

An A-IvIFS $\mathbb{A}_I$ in a non-empty universe χ is expressed as

$$\mathbb{A}_I=\{(x, M_{A_I}(x), N_{A_I}(x)) : x\in\chi, M_{A_I}(x) + N_{A_I}(x))\leq\mathbf{1}\}, \tag{62}$$

and the set of all A-IvIFSs is denoted by $\mathcal{A}_I$. Thus, an intuitionistic fuzzy truth value of an element in $\mathbb{A}_I$ is related to the ordered pair $(M_{A_I}(x), N_{A_I}(x))$.

When $\tilde{\mathbb{U}} = \{\widetilde{X} = (X_1, X_2) : (X_1, X_2) \in \mathbb{U}^2$ and $X_1 + X_2 \leq \mathbf{1}\}$[1] denotes the set of all Atanassov's interval-valued intuitionistic fuzzy degrees, and cosidering

$\mathbb{R}_I$**1:** $\tilde{X} \leq_{\tilde{\mathbb{U}}} \tilde{Y} \Leftrightarrow X_1 \leq Y_1$ and $X_2 \geq Y_2$;

$\mathbb{R}_I$**2:** $\tilde{X} \preceq_{\tilde{\mathbb{U}}} \tilde{Y} \Rightarrow X_1 \leq Y_1$ and $X_2 \leq Y_2$, for all $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{U}}$;

we have that $(\tilde{\mathbb{U}}, \leq_{\tilde{\mathbb{U}}})$ and $(\tilde{\mathbb{U}}, \preceq_{\tilde{\mathbb{U}}})$ are partial ordered sets with $\tilde{\mathbf{0}} = (\mathbf{0}, \mathbf{1}) \leq_{\tilde{\mathbb{U}}} \tilde{X}$ and $\tilde{\mathbf{1}} = (\mathbf{1}, \mathbf{0}) \geq_{\tilde{\mathbb{U}}} \tilde{X}$ as the least and greatest elements on $\tilde{\mathbb{U}}$, respectivelly.

An Atanassov's interval-valued intuitionistic fuzzy degree has the projections $l_{\mathbb{I}_\mathbb{I}}, r_{\mathbb{I}_\mathbb{I}} : \tilde{\mathbb{U}} \rightarrow \mathbb{U}$ defined by

$$l_{\mathbb{I}_\mathbb{I}}(\tilde{X}) = X_1 \text{ and } r_{\mathbb{I}_\mathbb{I}}(\tilde{X}) = X_2.$$

When $X_1 + X_2 = \mathbf{1}$ then A_I is restricted to the set $\mathcal{A}$ of all interval-valued fuzzy sets.

A function $\pi_{A_I} : \chi \rightarrow \mathbb{U}$, called an interval-valued intuitionistic fuzzy index (A-IvIFIx) of an element $x \in \chi$, related to an A-IvIFS A_I, is given as

$$\pi_{A_I}(x) = \mathbb{N}_S(M_{A_I}(x) + N_{A_I}(x)), \tag{63}$$

modelling not only the uncertainty degree but also the hesitancy (indeterminance) degree of x in A_I.

Thus, the accuracy function $h_{A_I} : \chi \rightarrow \mathbb{U}$ provides the interval-valued accuracy degree of x in A_I, given as $h_{A_I}(x) + \pi_{A_I}(x) = \mathbf{1}$. So, it means that the largest $\pi_A(x)(h_A(x))$ the higher the hesitancy (accuracy) degree of $x \in A_I$.

Moreover, the difference between A_I and B_I is given by:

$$A_I - B_I = \{\tilde{X} = (\min(N_{A_I}(x), N_{B_I}(x)), \max(N_{A_I}(x), M_{B_I}(x))) : \tilde{X} \in \tilde{\mathbb{U}}, x \in \chi\}.$$

5.2.2 Interval-Valued Intuitionistic Conjugate Operator

A bijective and monotonic function $\Phi : \tilde{\mathbb{U}} \rightarrow \tilde{\mathbb{U}}$ is an interval-valued intuitionistic fuzzy automorphism (IvIFA) on $\tilde{\mathbb{U}}$, meaning that below properties hold:

[1] We assume the componentwise addition on $\mathbb{U}$, see (MOORE, 1979).

$\mathbb{A}_I1$**:** $\Phi(\tilde{\mathbf{1}}) = \tilde{\mathbf{1}}$ and $\Phi(\tilde{\mathbf{0}}) = \tilde{\mathbf{0}}$;

$\mathbb{A}_I2$**:** $\Phi \circ \Phi^{-1}(\tilde{X}) = \tilde{X}$;

$\mathbb{A}_I3$**:** $\tilde{X} \leq_{\tilde{\mathbb{U}}} \tilde{Y}$ if and only if $\Phi(\tilde{X}) \leq_{\tilde{\mathbb{U}}} \Phi(\tilde{Y})$, for all $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{U}}$.

In the set of all IvIFAs ($Aut(\tilde{\mathbb{U}})$), the conjugate function of $f_I : \tilde{\mathbb{U}}^n \to \tilde{\mathbb{U}}$ is a function $f_I^{\Phi} : \tilde{\mathbb{U}}^n \to \tilde{\mathbb{U}}$, defined as follows

$$f_I^{\Phi}(\tilde{\mathbf{X}}) = \Phi^{-1}(f_I(\Phi(\tilde{X}_1), \ldots, \Phi(\tilde{X}_n))). \quad (64)$$

Reporting main results in (COSTA; BEDREGAL; NETO, 2011, Theorem 17), let $\phi : \mathbb{U} \to \mathbb{U}$ be an interval-valued automorphism, $\phi \in Aut(\mathbb{U})$. Then, a ϕ-representability of Φ is given by

$$\Phi(\tilde{X}) \;=\; (\phi(l_{\tilde{\mathbb{U}}}(\tilde{X})), \mathbf{1} - \phi(\mathbf{1} - r_{\tilde{\mathbb{U}}}(\tilde{X}))), \forall \tilde{X} \in \tilde{\mathbb{U}}; \quad (65)$$

Moreover, when $\phi \in Aut(U)$, for all $\tilde{X} \in \tilde{\mathbb{U}}$, a ϕ_U-representability of Φ is given by

$$\Phi(\tilde{X}) \;=\; \left([\phi_U(\underline{X}_1), \phi_U(\overline{X}_1)], [1 - \phi_U(1 - \underline{X}_2, 1 - \phi_U(1 - \overline{X}_2]\right). \quad (66)$$

5.2.3 Interval-Valued Intuitionistic Dual Conectives

An interval-valued intuitionistic fuzzy negation (IvIFN shortly) $\mathbb{N}_I : \tilde{\mathbb{U}} \to \tilde{\mathbb{U}}$ satisfies, for all $\tilde{X}, \tilde{Y} \in \tilde{\mathbb{U}}$, the following properties:

$\mathbb{N}_I$**1:** $\mathbb{N}_I(\tilde{\mathbf{0}}) = \mathbb{N}_I(\mathbf{0}, \mathbf{1}) = \tilde{\mathbf{1}}$ and $\mathbb{N}_I(\tilde{\mathbf{1}}) = \mathbb{N}_I(\mathbf{1}, \mathbf{0}) = \tilde{\mathbf{0}}$;

$\mathbb{N}_I$**2:** If $\tilde{X} \geq_{\tilde{\mathbb{U}}} \tilde{Y}$ then $\mathbb{N}_I(\tilde{x}) \leq_{\tilde{\mathbb{U}}} \mathbb{N}_I(\tilde{y})$.

Moreover, $\mathbb{N}_I$ is a strong IvIFN verifying the condition:

$\mathbb{N}_I3$**:** $\mathbb{N}_I(\mathbb{N}_I(\tilde{X})) = \tilde{X}, \forall \tilde{X} \in \tilde{\mathbb{U}}$.

Consider $\mathbb{N}_I$ as IvIFN and $\tilde{f} : \tilde{\mathbb{U}}^n \to \tilde{\mathbb{U}}$. The $\mathbb{N}_I$-dual interval-valued intuitionistic function of $\tilde{f}$, denoted by $\tilde{f}_{\mathbb{N}_I} : \tilde{\mathbb{U}}^n \to \tilde{\mathbb{U}}$, is given by:

$$\tilde{f}_{\mathbb{N}_I}(\tilde{\mathbf{X}}) = \mathbb{N}_I(\tilde{f}(\mathbb{N}_I(\tilde{X}_1), \ldots, \mathbb{N}_I(\tilde{X}_n))), \forall \tilde{\mathbf{X}} = (\tilde{X}_1, \ldots, \tilde{X}_n) \in \tilde{\mathbb{U}}^n. \quad (67)$$

When $\tilde{\mathbb{N}}_I$ is a strong IvIFN, $\tilde{f}$ is a self-dual interval-valued intuitionistic function. And, by (BACZYNSKI, 2004), taking a strong IvFN $\mathbb{N} : \mathbb{U} \to \mathbb{U}$, a IvIFN $\mathbb{N}_I : \tilde{\mathbb{U}} \to \tilde{\mathbb{U}}$ such that

$$\mathbb{N}_I(\tilde{X}) = (\mathbb{N}(\mathbb{N}_S(X_2)), \mathbb{N}_S(\mathbb{N}(X_1))), \quad (68)$$

is a strong IvIFN generated by means of the standard IvFN $\mathbb{N}_S$. Additionally, if $\mathbb{N} = \mathbb{N}_S$, Eq. (31) can be reduced to

$$\mathbb{N}_I(\tilde{X}) = (X_2, X_1) = [N_S(\overline{X}), N_S(\underline{X})].$$

Concluding this section, the complement of A-IvIFS A_I is defined by

$$\mathbb{A}_{IC}=\{(x, N_{A_I}(x), M_{A_I}(x)) : x\in\chi, M_{A_I}(x) + N_{A_I}(x))\leq\mathbf{1}\}, \quad (69)$$

6 INTERVAL EXTENSION OF THE GENERALIZED ATANASSOV'S INTUITIONISTIC FUZZY INDEX

Since Atanassov's interval-valued intuitinistic fuzzy logic was introduced, many researchers have taken advantage of interval-valued intuitionistic fuzzy index to represent not only the incertaitnty but also the imprecision in modelling the membership and non-membership functions, which is strictly linked by interval-valued fuzzy connectives and relevant in the composition of if-then rule of corresponding fuzzy system.

In intuitionistic fuzzy reasoning theory, intuitionistic fuzzy index operators play an important role. In this chapter we introduce distinct expressions for IvIFIx operators which can be used in real world applications, investigating properties, dual and conjugate constructions.

Focusing on the expressions of Atanassov's interval-valued intuitionistic fuzzy index based on the use of interval-valued fuzzy (co)implications, a methodology to provide new expressions that preserve such properties are considered.

6.1 Generalized Atanassov's Interval-valued Intuitionistic Fuzzy Index

Denoting a measure of non-determinacy, the intuitionistic fuzzy index of an element $x \in \chi$ in an interval-valued intuitionistic set $\mathbb{A}_I$, is conceived

In this section, we firstly introduced the axiomatic definition of a generalized interval-valued intuitionistic fuzzy index. In the sequence, its main properties and relationship with dual and conjugate operators are also discussed.

Definition 10. *A function $\Pi : \tilde{\mathbb{U}} \to \mathbb{U}$ is called a generalized Atanasso's interval-valued intuitionistic fuzzy index associated with a strong IvFN $\mathbb{N}$ (A-GIvIFIx($\mathbb{N}$)) if, for all $X_1, X_2, Y_1, Y_2 \in \mathbb{U}$, it holds that:*

$\Pi 1$: $\Pi(X_1, X_2) = \mathbf{1}$ *if and only if* $X_1 = X_2 = \mathbf{0}$*;*

$\Pi 2$: $\Pi(X_1, X_2) = \mathbf{0}$ *if and only if* $X_1 + X_2 = \mathbf{1}$*;*

$\Pi 3$**:** *If* $(Y_1, Y_2) \preceq_{\tilde{\mathbb{U}}} (X_1, X_2)$ *then* $\Pi(X_1, X_2) \leq_{\mathbb{U}} \Pi(Y_1, Y_2)$*;*

$\Pi 4$**:** $\Pi(X_1, X_2) = \Pi(\mathbb{N}_I(X_1, X_2))$ *when* $\mathbb{N}$ *is a SIvFN.*

6.2 Relationship with Interval-valued Fuzzy Connnectives

In this section, we discuss the condition under which an interval-valued fuzzy (co)implication gives rise to generalized Atanasso's interval-valued intuitionistic fuzzy index associated with a strong IvFN, describing a new methodology to obtain different expressions of such operator by making use of fuzzy (co)implication operators and their dual constructors.

In the following, Theorem 5 extends main results in (BARRENECHEA et al., 2009).

Theorem 5. *A function* $\Pi_{\mathbb{N},\mathbb{I}}(\Pi_{\mathbb{N},\mathbb{J}}) : \tilde{\mathbb{U}} \to U$ *is A-GIvIFIx(*$\mathbb{N}$*) iff exists a (co)implicator* $\mathbb{I}(\mathbb{J}) : U^2 \to U$ *verifying* $\mathbb{I}1$*(*$\mathbb{J}1$*),* $\mathbb{I}8$*(*$\mathbb{J}8$*),* $\mathbb{I}9$*(*$\mathbb{J}9$*) and* $\mathbb{I}10$*(*$\mathbb{J}10$*) such that*

$$\Pi_{\mathbb{I}}(X) = \mathbb{N}(\mathbb{I}(\mathbb{N}_{\mathbb{S}}(X_2), X_1)) \qquad (\Pi_{\mathbb{J}}(X) = \mathbb{J}(\mathbb{N}(\mathbb{N}_S(X_2)), \mathbb{N}(X_1))) \,. \tag{70}$$

Proof. The Eq.(70b) is proved below. Analogously, it can be done to prove Eq.(70a). ($\Rightarrow$) When $\mathbb{N}$ is involutive, $\mathbb{J} : \mathbb{U}^2 \to \mathbb{U}$ verifies $\mathbb{J}2$, $\mathbb{J}8$, $\mathbb{J}9$ and $\mathbb{J}10$, it holds that:

$$\begin{aligned}
\Pi_1 : \Pi_{\mathbb{N},\mathbb{J}}(X_1, X_2) = \mathbf{1} &\Leftrightarrow \mathbb{J}(\mathbb{N}(\mathbb{N}_S(X_2)), \mathbb{N}(X_1)) = \mathbf{1} \text{ (by Eq.(70b)}\\
&\Leftrightarrow \mathbb{N}_S(X_2) = \mathbf{1}\, and\, \mathbb{N}(X_1) = \mathbf{1} \text{ (by } \mathbb{J}2)\\
&\Leftrightarrow X_2 = X_1 = \mathbf{0} \text{ (by } \mathbb{N}1)\\
\Pi_2 : \Pi_{\mathbb{N},\mathbb{J}}(X_1, X_2) = \mathbf{0} &\Leftrightarrow \mathbb{J}(\mathbb{N}(1 - X_2), \mathbb{N}(X_1)) = \mathbf{0} \text{ (by Eq.(70b)}\\
&\Leftrightarrow \mathbb{N}(1 - X_2) \geq \mathbb{N}(X_1)\\
&\Leftrightarrow X_1 + X_2 \leq \mathbf{1} \text{ and } X_1 + X_2 \geq \mathbf{1}\\
&\Leftrightarrow X_1 + X_2 = \mathbf{1} \text{ (by } \mathbb{J}8 \text{ and Eq.(62))}\\
\Pi_3 : (Y_1, Y_2) \preceq (X_1, X_2) &\Rightarrow Y_1 \leq X_1\, and\, Y_2 \leq X_2 \text{ by } \mathbb{R}_I 2\\
&\Rightarrow \mathbb{N}(X_1) \geq \mathbb{N}(Y_1) and \mathbb{N}(1 - X_2) \leq \mathbb{N}(1 - Y_2) \text{ by } \mathbb{N}2\\
&\Rightarrow \mathbb{J}(\mathbb{N}(1 - X_2), \mathbb{N}(X_1)) \leq \mathbb{J}(\mathbb{N}(1 - Y_2), \mathbb{N}(Y_1)) \text{ by } \mathbb{J}1\\
&\Rightarrow \Pi_{\mathbb{N},\mathbb{J}}(X_1, X_2) \leq \Pi_{\mathbb{N},\mathbb{J}}(Y_1, Y_2) \text{ by Eq.(33)}\\
\Pi_4 : \Pi_{\mathbb{N},\mathbb{I}}(\mathbb{N}((X_1, X_2)) &= \Pi_{\mathbb{J}(\mathbb{N}(\mathbb{N}_{\mathbb{S}}(X_2)), \mathbb{N}_{\mathbb{S}}(\mathbb{N}(\mathbb{X}_{\not{\kappa}})))} \text{ by Eq.(31)}\\
&= (\mathbb{J}(X_1, 1 - X_2)) \text{ by Eq.(33)}\\
&= (\mathbb{J}(\mathbb{N}(1 - X_2)), \mathbb{N}(X_1)) \text{ by } \mathbb{J}9\\
&= \Pi_{\mathbb{N},\mathbb{J}}((X_1, X_2)(X_1, X_2)) \text{ by Eq.(33)}
\end{aligned}$$

($\Leftarrow$) Considering the function $J : \mathbb{U}^2 \to \mathbb{U}$ given as $\mathbb{J}(X_1, X_2) = \mathbf{1}$, if $X_1 > X_2$; and

$\mathbb{J}(X_1, X_2) = \Pi_{\mathbb{N},\mathbb{J}}(X_2, \mathbb{N}_S(\mathbb{N}(X_1)))$, otherwise, the following holds:

$$\mathbb{J}2: Y_1 \geq Y_2 \Leftrightarrow \mathbb{J}(X, Y_2) = \begin{cases} \mathbf{1}, \text{if } X > Y_1, \\ \Pi_{\mathbb{N},\mathbb{J}}(Y_2, \mathbb{N}_S(\mathbb{N}(X))), \quad \text{otherwise; by Eq.(33)} \end{cases}$$

$$\geq \begin{cases} \mathbf{1}, \text{if } X > Y_2, \\ \mathbb{J}(X, Y_2), \text{otherwise; by } \Pi_3 \text{ and Eq.(33).} \end{cases}$$

$\mathbb{J}8$: Strainghforward.

$$\mathbb{J}9: \mathbb{J}(\mathbb{N}(X_2), \mathbb{N}(X_1)) = \begin{cases} \mathbf{1}, \text{if } \mathbb{N}(X_2) > \mathbb{N}(X_1), \\ \Pi_{\mathbb{N},\mathbb{J}}(X_1, 1 - X_2), \text{otherwise;} \quad \text{by Eqs.(33) and (31)} \end{cases}$$

$$= \begin{cases} \mathbf{1}, \text{ if } X_1 \geq X_2, \\ \Pi_{\mathbb{N},\mathbb{J}}(\mathbb{N}(\mathbb{N}(X_2), \mathbb{N}_S(\mathbb{N}(X_1)))), \text{otherwise} \quad \text{by } \Pi_4 \text{ and Eq.(33)} \end{cases}$$

$$= \mathbb{J}(X_1, X_2), \text{whenever } \mathbb{N} \text{ is a SFN.}$$

$$\mathbb{J}10: \mathbb{J}(X_1, X_2) = \mathbf{1} \Leftrightarrow \Pi_{\mathbb{N},\mathbb{J}}(\mathbb{N}(X_2), \mathbf{1} - \mathbb{N}(X_1)) = \mathbf{1} \text{by Eq.(33)}$$

$$\Leftrightarrow \mathbb{N}(X_2) = \mathbf{1} - \mathbb{N}(X_1) = \mathbf{0} \Leftrightarrow X_1 = \mathbf{0}\, and\, X_2 = \mathbf{1} \text{ by } \Pi_1.$$

Therefore, Theorem 5 holds. □

6.2.1 Dual Operators and A-IvIFIx with respect to IvFN

The Φ-representability and $\mathbb{N}$-dual IvIFIx constructions are discussed below.

Proposition 19. *Let* $\mathbb{I}_{\mathbb{N}}$ *(*$\mathbb{J}_{\mathbb{N}}$*)* $: U^2 \to U$ *be the* $\mathbb{N}$*-dual operator of a (co)implication* $\mathbb{I}(\mathbb{J})$: $\mathbb{U}^2 \to \mathbb{U}$*. The following holds:*

$$\Pi_{\mathbb{I}_{\mathbb{N}}}(\tilde{\mathbb{X}}) = \Pi_{\mathbb{I}}(\tilde{\mathbb{X}}), \qquad \left(\Pi_{\mathbb{J}_{\mathbb{N}}}(\tilde{\mathbb{X}}) = \Pi_{\mathbb{J}}(\tilde{\mathbb{X}})\right). \tag{71}$$

Proof. $\Pi_{\mathbb{N},\mathbb{I}_{\mathbb{N}}}(\tilde{\mathbb{X}}) = \mathbb{N}(\mathbb{I}(\mathbb{N}_S(X_2), X_1)) = \mathbb{I}_{\mathbb{N}}(\mathbb{N}(\mathbb{N}_S(X_2)), \mathbb{N}(X_1)) = \Pi_{\mathbb{N},\mathbb{I}}(\tilde{\mathbb{X}})$, $\forall \tilde{\mathbb{X}} \in \tilde{\mathbb{U}}$. □

In diagram of Figure 7 the following denotation is considered:
(i) $C(\mathbb{I})$ and $C(\mathbb{J})$ denote the class of all (co)implications;
(ii) $C(\mathbb{N})$ denotes the class of all negations;
(iii) $C(\Pi)$ provides denotation to the class of all A-IFIx;
In addition, the interrelations summarize the results stated in Theorem 5 and Proposition 19 are summarized in the diagram presented in Figure 7).

Corollary 2. *When* $\mathbb{N} = \mathbb{N}_S$*, Eq.(70) in Theorem 5 is given as*

$$\Pi_{\mathbb{I}}(\tilde{X}) = \mathbb{N}_S(\mathbb{I}(\mathbb{N}_S(X_2), X_1)) \qquad \left(\Pi_{\mathbb{J}}(\tilde{X}) = \mathbb{J}(X_2, \mathbb{N}_S(X_1)\right). \tag{72}$$

Proposition 20. *Let* $\mathbb{N}$ *be an* N*-representable strong IvFN and* $\pi_{N,I} : \tilde{U} \to U$ *be A-IFIx(N). If* $\mathbb{I}, \mathbb{J}$ *are representable (co)implications given by Eq.(61), a function* $\Pi_{\mathbb{I}}$:

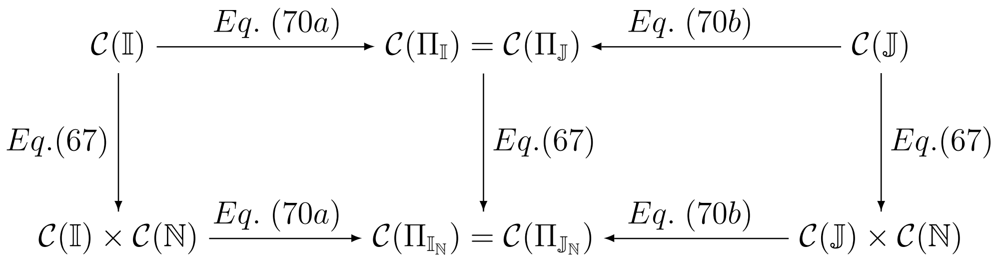

Figure 7 – Constructing A-IvIFIx from classes of implications.

$\tilde{\mathbb{U}} \to \mathbb{U}$ *given by Eq.(72) can be expressed as*

$$\Pi_{\mathbb{I}}(\tilde{X}) = [\Pi_{N,I}(\overline{X}_2, \overline{X}_1), \Pi_I(\underline{X}_2, \underline{X}_1)] \tag{73}$$

$$\Pi_{\mathbb{J}}(\tilde{X}) = [\Pi_J(\overline{X}_2, \overline{X}_1), \Pi_J(\underline{X}_2, \underline{X}_1)]. \tag{74}$$

Proof. We prove Eq.(73a), the other one can be analogously done. By taking $X_1 = [\underline{X}_1, \overline{X}_1]$ and $X_2 = [\underline{X}_2, \overline{X}_2]$ then $X_1 + X_2 = [\underline{X}_1 + \underline{X}_2, \overline{X}_1 + \overline{X}_2] \leq [1,1]$, meaning that $\overline{X}_1 + \overline{X}_2 \leq 1$ and $\underline{X}_1 + \underline{X}_2 \leq 1$. Therefore $\Pi_{\mathbb{N},\mathbb{I}}(\tilde{X}) = \mathbb{N}(\mathbb{I}([1 - \overline{X}_2, 1 - \underline{X}_1], [\underline{X}_1, \overline{X}_1])) = [N(I(1 - \overline{X}_2, \overline{X}_1)), N(I(1 - \underline{X}_2, \underline{X}_1))]$. Concluding, $\Pi_{\mathbb{N},\mathbb{I}}(\tilde{X}) = [\Pi_{N,I}(\overline{X}_2, \overline{X}_1), \Pi_{N,I}(\underline{X}_2, \underline{X}_1)]$. So, Proposition 20 holds. □

Example 8. *Consider* $\mathbb{I}_{\mathbb{RC}}$ *and related* $\mathbb{N}_{\mathbb{S}}$*-dual construction* $\Pi_{\mathbb{N}_{\mathbb{S}},\mathbb{J}_{\mathbb{RC}}}$*. By preserving the conditions of Proposition 20, Eq.(33) can be expressed as*

$$\Pi_{\mathbb{N}_{\mathbb{S}},\mathbb{I}_{\mathbb{RC}}}(X_1, X_2) = \Pi_{\mathbb{N}_{\mathbb{S}},\mathbb{J}_{\mathbb{RC}}}(X_1, X_2)$$

$$\Pi_{\mathbb{N}_{\mathbb{S}},\mathbb{I}_{\mathbb{RC}}}(X_1, X_2) = \begin{cases} 0, \text{ if } X_1 + X_2 = 1, \\ 1 - [1 - \overline{X}_2 - \overline{X}_1 + \overline{X}_2\overline{X}_1, 1 - \underline{X}_2 - \underline{X}_1 + \underline{X}_2\underline{X}_1], \text{ otherwise}; \end{cases} \tag{75}$$

In analogous way, the methodology can be applied to others implications, obtaining other examples of generalized interval-valued intuitionistic fuzzy indexes associated with the interval extension of the standard negation.

Table 14, in the following, presents the method applied to $\mathbb{I}_{LK}$, $\mathbb{I}_{GR}$ and $\mathbb{I}_{30}$.

6.2.1.1 Comparing results from application of the constructive method related to A-GIvIFIx

According with (BACZYNSKI, 2004), the four fuzzy implications in Table 14 can be ordered as follows:

$$\Pi_{GR}(X,Y) \geq \Pi_{LK}(X,Y) \geq \Pi_{RB}(X,Y) \geq \Pi_{KD}(X,Y) \tag{76}$$

However, the other implication $\Pi_{30}(X,Y)$ is not comparable with the above four

Table 14 – Generalized interval valued intuitionistic fuzzy index associated with the standard negation.

Dual Functions Fuzzy	$A-GIvIFIx(N_{SI})$
$\mathbb{I}_{KD}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X\leq Y,\\ \max(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$ $\mathbb{J}_{KD}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X\geq Y,\\ \min(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$	$\Pi_{KD}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X+Y=\mathbf{1},\\ \mathbf{1}-\max(X,Y), & \text{otherwise};\end{cases}$
$\mathbb{I}_{LK}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X\leq Y,\\ \mathbf{1}-X+Y, & \text{otherwise};\end{cases}$ $\mathbb{J}_{LK}(x,y)=\begin{cases}\mathbf{0}, & \text{if } X\geq Y,\\ Y-X, & \text{otherwise};\end{cases}$	$\Pi_{LK}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X+Y=\mathbf{1},\\ \mathbf{1}-X-Y, & \text{otherwise};\end{cases}$
$\mathbb{I}_{RB}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X\leq Y,\\ \mathbf{1}-X+XY, & \text{otherwise};\end{cases}$ $\mathbb{J}_{RB}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X\geq Y,\\ Y-XY, & \text{otherwise};\end{cases}$	$\Pi_{RB}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X+Y=\mathbf{1},\\ \mathbf{1}-X-Y+XY, & \text{otherwise};\end{cases}$
$\mathbb{I}_{GR}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X\leq Y,\\ \mathbf{0}, & \text{otherwise};\end{cases}$ $\mathbb{J}_{GR}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X\geq Y,\\ \mathbf{1}, & \text{otherwise};\end{cases}$	$\Pi_{GR}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X+Y=\mathbf{1},\\ \mathbf{1}, & \text{otherwise};\end{cases}$
$\mathbb{I}_{30}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X\leq Y,\\ \max(\mathbf{1}-X,Y,\mathbf{0.5}), & \text{if } 0<Y<X<1;\\ \max(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$ $\mathbb{J}_{30}(X,Y)=\begin{cases}\mathbf{0}, & \text{if } X\geq Y,\\ \min(\mathbf{1}-X,Y,\mathbf{0.5}), & \text{if } \mathbf{0}<Y<X<\mathbf{1};\\ \min(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$	$\Pi_{30}(X,Y)=\begin{cases}\mathbf{1}, & \text{if } X\leq Y\\ \min(\mathbf{1}-X,Y,\mathbf{0.5}), & \text{if } \mathbf{0}<Y<X<\mathbf{1};\\ \min(\mathbf{1}-X,Y), & \text{otherwise};\end{cases}$

implications, impling the use of admissible linear orders to order all these operators.

Example 9. *Consider the following A-IvIFSs:*

$$\begin{aligned} A_1 &= \{(u,[0.3,0.4],[0.1,0.2])\}\\ A_2 &= \{(u,[0.5,0.7],[0.2,0.4])\}\\ A_3 &= \{(u,[0.3,0.4],[0.0,0.0])\}\end{aligned}$$

Based on A-IvIFIx, obtained through of interval-valued implications $\mathbb{I}_{LK},\mathbb{I}_{RB},\mathbb{I}_{30}$ *the following interval-values holds:*

	A_1	A_2	A_3
$\Pi_{\mathbb{I}_{\mathbb{LK}}}$	[0.4, 0.6]	[0, 0]	[0.6, 0.7]
$\Pi_{\mathbb{I}_{\mathbb{RB}}}$	[0.48, 0.72]	[0.18, 0.4]	[0.6, 0.7]
$\Pi_{\mathbb{I}_{30}}$	[0.5, 0.5]	[0.3, 0.5]	[0.6, 0.7]

Using the lexicographical linear order $\sqsubseteq_{Lex1}, \sqsubseteq_{Lex2}$, by Definition 5, we have that:

$$\Pi_{\mathbb{I}_{\mathbb{LK}}}(X,Y) \leq \Pi_{\mathbb{I}_{\mathbb{RB}}}(X,Y) \leq \Pi_{\mathbb{I}_{30}}(X,Y)$$

Analyzing the same input set, with the implication of $\mathbb{I}_{\mathbb{GR}}$, we can infer that $\mathbb{I}_{\mathbb{GR}}(X,Y) \geq \mathbb{I}_{\mathbb{LK}}(X,Y) \geq \mathbb{I}_{\mathbb{RB}}(X,Y)$ than we conclude that

$$\Pi_{\mathbb{I}_{\mathbb{GR}}}(X,Y) \leq \Pi_{\mathbb{I}_{\mathbb{LK}}}(X,Y) \leq \Pi_{\mathbb{I}_{\mathbb{RB}}}(X,Y) \leq \Pi_{\mathbb{I}_{30}}(X,Y)$$

6.2.2 Relationship with Interval-valued Automorphisms

Proposition 21. *Let $\mathbb{N}^{\Phi} : \mathbb{U} \rightarrow \mathbb{U}$ be the ϕ-conjugate of a strong IvFN $\mathbb{N} : \mathbb{U} \rightarrow \mathbb{U}$ and $\phi : \mathbb{U} \rightarrow \mathbb{U}$ be a ϕ-representable IvA given by Eq.(66). When $\Phi : \tilde{\mathbb{U}} \rightarrow \tilde{\mathbb{U}}$ is a Φ-representable IvIF given by Eq.(65), a function $\Pi^{\Phi} : \tilde{\mathbb{U}} \rightarrow \mathbb{U}$ given by*

$$\Pi^{\Phi}(X_1, X_2) \;\; = \;\; (\phi^{-1}(\Pi(\phi(X_1)), \mathbf{1} - \phi(\mathbf{1} - X_2)), \tag{77}$$

is an A-GIvIFIx($\mathbb{N}_I$) whenever $\Pi : \tilde{\mathbb{U}} \rightarrow \mathbb{U}$ is also an A-GIvIFIx($\mathbb{N}_I$).

Proof. Let $\phi : \tilde{\mathbb{U}} \rightarrow \mathbb{U}$ be a ϕ-representable IvA and $\Pi : \tilde{\mathbb{U}} \rightarrow \mathbb{U}$ be an A-GIvIFIx(N_I). It holds that:

$\mathbf{\Pi_1}$:$\Pi^\Phi(X_1, X_2) = \mathbf{1} \Leftrightarrow \phi^{-1}(\Pi(\phi(X_1), \mathbf{1} - \phi(\mathbf{1} - X_2))) = \mathbf{1}$ (by Eq.(65))

$\Leftrightarrow \Pi(\phi(X_1), \mathbf{1} - \phi(\mathbf{1} - X_2)) = \mathbf{1}$ (by $\mathbb{A}_I 1$)

$\Leftrightarrow \phi(X_1) = \mathbf{0}\, and\, \mathbf{1} - \phi(\mathbf{1} - X_2) = \mathbf{0}$ (by $\mathbf{\Pi_1}$)

$\Leftrightarrow X_1 = \mathbf{0}\, and\, X_2 = \mathbf{0}$ (by $\mathbb{A}_I 1$)

$\mathbf{\Pi_2}$:$\Pi^\Phi(X_1, X_2) = 0 \Leftrightarrow \phi^{-1}(\Pi(\phi(X_1), \mathbf{1} - \phi(\mathbf{1} - X_2))) = \mathbf{0}$ (by Eq.(65))

$\Leftrightarrow \Pi(\phi(X_1), \mathbf{1} - \phi(\mathbf{1} - X_2)) = \mathbf{0}$ (by $\mathbb{A}_I 1$)

$\Leftrightarrow \phi(X_1) + \mathbf{1} - \phi(\mathbf{1} - X_2) = \mathbf{1}$ (by $\mathbf{\Pi_2}$)

$\Leftrightarrow \phi(X_1) = \phi(\mathbf{1} - X_2) \Leftrightarrow X_1 = \mathbf{1} - X_2$ or $\Leftrightarrow X_1 + X_2 = \mathbf{1}$ (by $\mathbb{A}_I 1$)

$\mathbf{\Pi_3}$:$(X_1, X_2) \preceq (Y_1, Y_2) \Rightarrow X_1 \leq Y_1\, and\, X_2 \leq Y_2$ by $\preceq$-relation

$\Rightarrow \phi(X_1) \leq \phi(Y_1)\, and\, \mathbf{1} - \phi(\mathbf{1} - X_2) \leq \mathbf{1} - \phi(\mathbf{1} - Y_2)$ by $\mathbb{A}_I 1$

$\Rightarrow \Pi(\phi(X_1), \mathbf{1} - \phi(\mathbf{1} - X_2)) \leq \Pi(\phi(Y_1), \mathbf{1} - \phi(1 - Y_2))$ by $\mathbf{\Pi_3}$

$\Rightarrow \phi^{-1}\left(\Pi(\phi(X_1), \mathbf{1} - \phi(1 - X_2))\right) \leq \phi^{-1}\left(\Pi(\phi(Y_1), \mathbf{1} - \phi(\mathbf{1} - Y_2))\right)$ by $\mathbb{A}_1$

$\Rightarrow \Pi^\phi_G(X_1, X_2) \leq \Pi^\phi_G(Y_1, Y_2)$ (by Eq.(65)).

Let $\mathbb{N}_I$ be a strong IvIFN given by Eq.(31) and $\mathbb{N}_I^\Phi$ its $\Phi-$conjugate function.

$$\mathbf{\Pi_4} : \Pi^\Phi\left(\mathbb{N}_I^\Phi(X_1, X_2)\right) = \phi^{-1}\left(\Pi(\phi \circ \phi^{-1}(\mathbb{N}_I(\phi(X_1, X_2))))\right) \text{ (by Eq. (28))}$$
$$= \phi^{-1}\left(\Pi(\mathbb{N}_I(\phi(X_1, X_2)))\right) = \phi^{-1}\left(\Pi(\phi(X_1, X_2))\right) = \Pi(X_1, X_2) \text{ (by } \mathbf{\Pi_4}\text{)}$$

Therefore, Proposition 21 holds. □

Corollary 3. *In conditions of Proposition 21 and considering a ϕ-representable IvA given by Eq.(66), we can express Eq.(40) as follows:*

$$\Pi^\Phi(X_1, X_2) \;=\; \left[\Pi^\phi(\underline{X_1}, \underline{X_2}), \Pi^\phi(\overline{X_1}, \overline{X_2})\right]. \tag{78}$$

Proof. Straightforward from Proposition 21. □

Corollary 4. *Let Φ be a ϕ-representable automorphism in $Aut(\tilde{\mathbb{U}})$ and $\mathbb{I}(\mathbb{J}) : \tilde{\mathbb{U}}^2 \to \tilde{\mathbb{U}}$ be the corresponding ϕ-conjugate operator related to a (co)implication $\mathbb{I}(\mathbb{J}) : \mathbb{U}^2 \to \mathbb{U}$, verifying the conditions of Theorem 5. When $\mathbb{N}^\Phi$ is a strong ϕ-conjugate IvFN negation, a function $\Pi_{\mathbb{I}^\phi}(\Pi_{\mathbb{J}^\phi}) : \tilde{\mathbb{U}} \to \mathbb{U}$ given by*

$$\Pi^\Phi_{\mathbb{I}}(X_1, X_2) = \mathbb{N}^\Phi(\mathbb{I}^\Phi(\mathbb{N}_S(X_2), X_1)) \tag{79}$$
$$\left(\Pi^\Phi_{\mathbb{J}}(X_1, X_2) = \mathbb{J}^\phi(\mathbb{N}^\Phi(\mathbb{N}_S(X_2), \mathbb{N}^\Phi(X_1))\right). \tag{80}$$

is an A-GIvIFIx(N).

Proof. It follows from Proposition 21 and Theorem 5. □

The main results in Corollary 4 and Proposition 21 are summarized in the diagram below (figure 8):

In diagram of Figure 8 the following denotation is considered:
(i) $C(\mathbb{I})$ and $C(\mathbb{J})$ denote the class of all (co)implications;
(ii) $C(\mathbb{N})$ denotes the class of all negations;
(iii) $Aut(\mathbb{U})$ denotes the class of all automorphisms;
(iv) $C(\Pi)$ provides denotation to the class of all A-GIvIFIx;
These interrelations summarize the results stated in Corollarys 3 and 4.

$$\begin{array}{ccc}
\mathcal{C}(\mathbb{I}) \times \mathcal{C}(\mathbb{N}) \times \mathcal{A}ut(\mathbb{U}) & \xrightarrow{Eq.(70)} & \mathcal{C}(\Pi_{\mathbb{I}}) \times \mathcal{A}ut(\mathbb{U}) \\
\Big\downarrow Eq.(64) & & \Big\downarrow Eq.(64) \\
\mathcal{C}(\mathbb{I}^{\Phi}) \times \mathcal{C}(\mathbb{N}^{\Phi}) & \xrightarrow{Eq.(70)} & \mathcal{C}(\Pi_{\mathbb{I}}^{\Phi}) = \mathcal{C}(\Pi_{\mathbb{I}^{\Phi}})
\end{array}$$

Figure 8 – Diagramatical expression in the classes of A-IvIFIx and $Aut(\mathbb{U})$

Example 10. *Consider $\mathbb{I}_{\mathbb{RC}}$ and related Φ-conjugate construction $\Pi^{\Phi}_{\mathbb{N}_{\mathbb{S}},\mathbb{J}_{\mathbb{RC}}}$ given by Eq.(75). For a ϕ-representable automorphim given as $\phi(X) = X^n$, whenever n is a nonnegative integer, we have the following:*

$$\Pi^{\Phi}{}_{\mathbb{N}_{\mathbb{S}},\mathbb{I}_{\mathbb{RC}}}(X_1, X_2) \;=\; \left[\sqrt[n]{(1-\overline{X}_1^n)(1-\overline{X}_2)^n};\ \sqrt[n]{(1-\underline{X}_1^n)(1-\underline{X}_2)^n}\right]. \qquad (81)$$

7 INTERVAL-VALUED INTUITIONISTIC FUZZY ENTROPY

The concept of entropy which measures the fuzziness of a fuzzy set was introduced by De Luca and Termini (LUCA; TERMINI, 1972) in order to measure how far a fuzzy set (FS) is from a crisp one. Since then, this concept has been adapted to the different extensions of FSs and with different interpretations, as in modelling type-2 fuzzy sets (MIGUEL et al., 2017) (XU; SHEN, 2014), interpreting vague sets (ZHANG; JIANG, 2008), dealing with intuitionistic fuzzy set (WEI; GAO; GUO, 2012), (YE, 2010),(VERMA; SHARMA, 2013), (LIU; REN, 2014) and also modelling interval-valued intuitionistic fuzzy sets (JING; MIN, 2013) (ZHANG; JIANG, 2008), all of them measure how far the considered extension is from a fuzzy set of reference.

In this sense, it is worth mentioning the following concepts: the Atanassov intuitionistic fuzzy entropy measure, given by Szmidt and Kacprzyk (SZMIDT; KACPRZYK, 2001) to measure how far A-IFS is from a crisp set. The entropy for interval-valued fuzzy sets (IVFSs) defined by Burillo and Bustince (BURILLO; BUSTINCE, 1996), which measures how far an IVFS or A- AIFS is from FS.

The generalized interval-valued intuitionistic fuzzy index seems to be suitable to deal with measures of entropy in A-IvIFSs, modelling uncertainty and imprecision in membership and non-membership functions.

Following this approach, this chapter generalizes results from (BUSTINCE et al., 2011), discussing properties related to Atanassov's interval-valued intuitionistic fuzzy entropy (A-IvIFE) which are obtained by action of an interval-valued aggregation applied to the generalized interval-valued intuitionistic fuzzy index.

Definition 11. *An interval-valued function $\mathbb{E} : \mathcal{A}_I \to \mathbb{U}$ is called an A-IvIFE if $\mathbb{E}$ verifies the following properties:*

$\mathbb{E}2$*: $\mathbb{E}(A_I) = \mathbf{0} \Leftrightarrow A_I \in \mathcal{A}$;*

$\mathbb{E}2$*: $\mathbb{E}(A_I) = \mathbf{1} \Leftrightarrow M_{A_I}(x) = N_{A_I}(x) = \mathbf{0}$, $\forall x \in \chi$;*

$\mathbb{E}3$*: $\mathbb{E}(A_I) = \mathbb{E}(A_{Ic})$;*

$\mathbb{E}4$*: If $A_I \preceq_{\tilde{\mathbb{U}}} B_I$ then $\mathbb{E}(A_I) \geq_{\mathbb{U}} \mathbb{E}(B_I)$, $\forall A_I, B_I \in \mathcal{A}_I$.*

Now, main properties of A-IvIFE obtained by A-GIvIFIx are studied.

Theorem 6. *Consider* $\chi = \{x_1, \ldots, x_n\}$. *Let* $\mathbb{M} : \mathbb{U}^n \to \mathbb{U}$ *be an interval-valued aggregation,* $\mathbb{N}$ *be a strong IvFN and* Π *be an* $A - GIvIFIx(\mathbb{N})$. *A function* $\mathbb{E} : \mathcal{A}_I \to \mathbb{U}$ *given by*

$$\mathbb{E}(A_I) = \mathbb{M}_{i=1}^n \Pi(A_I(x_i)), \forall x_i \in \chi, \tag{82}$$

is an A-IvIFE in the sense of Definition 11.

Proof. Let A_{I_c} be the complement of A_I given by Eq.(69). For all $x_i \in \chi$ and $A_I, B_I \in \mathcal{A}_I$, we have that:

$\mathbb{E}1$: $\mathbb{E}(A_I) = \mathbf{0} \Leftrightarrow \mathbb{M}_{i=1}^n \Pi(A_I(x_i)) = \mathbf{0}$. By $\mathbb{M}1$, $\mathbb{E}(A_I) = \mathbf{0} \Leftrightarrow M_{A_I}(x_i) + N_{A_I}(x_i) = \mathbf{1}$. Then, by $\Pi 2$, $\mathbb{E}(A_I) = \mathbf{0} \Leftrightarrow A_I \in \mathcal{A}$.

$\mathbb{E}2$: $\mathbb{E}(A_I) = \mathbf{1} \Leftrightarrow \mathbb{M}_{i=1}^n \Pi(A_I(x_i)) = \mathbf{1}$. By $\mathbb{M}1$, $\mathbb{E}(A_I) = \mathbf{1} \Leftrightarrow M_{A_I}(x_i) + N_{A_I}(x_i) = \mathbf{0}$, meaning that $M_{A_I}(x_i) = N_{A_I}(x_i) = \mathbf{0}$.

$\mathbb{E}3$: $\mathbb{E}(A_I)_c = \mathbb{M}_{i=1}^n \Pi(A_{Ic}(x_i)) = \Pi(\mathbb{N}_I(X_1, X_2))$. By $\Pi 3$, the following holds $\mathbb{E}(A_I)_c = \Pi(X_1, X_2)$. Concluding, $\mathbb{E}(A_I)_c = \mathbb{E}(A_I)$.

$\mathbb{E}4$: If $A_I \preceq_{\tilde{\mathbb{U}}} B_I$ then $A_I(x_i) \preceq_{\tilde{\mathbb{U}}} B_I(x_i)$. Based on $\Pi 3$, it holds that $\Pi(B_I(x_i)) \leq_{\mathbb{U}} \Pi(A_I(x_i))$. By $\mathbb{M}3$, we obtain that $\mathbb{M}_{i=1}^n \Pi(B_I(x_i)) \leq_{\mathbb{U}} \mathbb{M}_{i=1}^n \Pi(B_I(x_i))$. As conclusion, $\mathbb{E}(A_I) \geq_{\mathbb{U}} \mathbb{E}(B_I)$.

Therefore, Theorem 6 is verified. □

The next proposition formalizes the interval extension of the constructive method to obtain interval fuzzy entropy from results in (BUSTINCE et al., 2011).

Proposition 22. *Consider* $\chi = \{x_1, \ldots, x_n\}$. *Let* $\mathbb{M} : \mathbb{U}^n \to \mathbb{U}$ *be an interval-valued aggregation,* $\mathbb{N}$ *be a strong IvFN and* $\Pi_{\mathbb{N},\mathbb{I}}(\Pi_{\mathbb{N},\mathbb{J}}) : \tilde{\mathbb{U}} \to U$ *is A-GIvIFIx(*$\mathbb{N}$*) given by Eq.(70). Then, for all* $x_i \in \chi$*, an A-IvIFE* $\mathbb{E} : \mathcal{A}_I \to \mathbb{U}$ *can be given by*

$$\mathbb{E}_{\Pi_{\mathbb{I}}}(A_I) = \mathbb{M}_{i=1}^n \Pi_{\mathbb{I}}(A_I(x_i)); \tag{83}$$

$$\mathbb{E}_{\Pi_{\mathbb{I}}}(A_I) = \mathbb{M}_{i=1}^n \Pi_{\mathbb{J}}(A_I(x_i)). \tag{84}$$

Proof. Straightforward from Theorems 5 and 6. □

Corollary 5. *Consider* $\mathbb{N} = \mathbb{N}_S$*, A-GIvIFIx (*$\mathbb{N}_S$*)* $\Pi_{\mathbb{N},\mathbb{I}}$ *given by Eq.(70). Then, by taking* $A_I(x_i) = (M_{A_I}(x_i) = X_{1i}, N_{A_I}(x_i) = X_{2i})$ *for all* $x_i \in \chi$*, an A-IvIFE* $\mathbb{E} : \mathcal{A}_I \to \mathbb{U}$ *which is given in Eq.(82) can be expressed as*

$$\mathbb{E}_{\Pi_{\mathbb{I}}}(A_I(x_i)) = \mathbb{M}_{i=1}^n \left(\mathbb{N}_S(\mathbb{I}(\mathbb{N}_S(X_{2i}), X_{1i})\right); \tag{85}$$

$$\mathbb{E}_{\Pi_{\mathbb{J}}}(A_I(x_i)) = \mathbb{M}_{i=1}^n \mathbb{J}(X_{2i}, \mathbb{N}_S(X_{1i}). \tag{86}$$

Proof. Straightforward from Proposition 22 and Theorem 5. □

Example 11. *Consider the arithmetic mean as an aggregation operator,* $\mathbb{N}_S$ *as the IvFN and* $\mathbb{I}_{\mathbb{RC}}$ *related to the A-GIvIFIx given in Eq.(75). For all* $x_i \in \chi$ *and* $(X_{1i}, X_{2i}) \in \tilde{\mathbb{U}}$ *defining an A-IvIFS* A_I*, we have the following:*

$$\begin{aligned} \mathbb{E}_{\Pi_{\mathbb{N}_S,\mathbb{I}_{RB}}}(X_{1i}, X_{2i}) &= \frac{1}{n}\sum_{i=1}^{n}(\mathbb{N}_S(\mathbb{I}_{RB}(\mathbb{N}_S(X_{2i}), X_{1i})) \\ &= \frac{1}{n}\sum_{i=1}^{n}[1-\overline{X}_{2i}-\overline{X}_{1i}+\overline{X}_{2i}\overline{X}_{1i}, 1-\underline{X}_{2i}-\underline{X}_{1i}+\underline{X}_{2i}\underline{X}_{1i}] \end{aligned} \tag{87}$$

7.0.1 Relationship with Intuitionistic Index and Conjugate Operators

Conjugation and duality properties related to generalized Atanassov's Intuitionistic Fuzzy Index are reported from (COSTA et al., 2017).

Proposition 23. *Consider* $\chi = \{x_1, \ldots, x_n\}$ *and* $\Phi \in Aut(\tilde{\mathbb{U}})$ *a* ϕ*-representable IvIFA given by Eq.(65). When* Π *is A-GIvIFIx(*$\mathbb{N}$*), an A-IvIFE is a function* $\mathbb{E}^{\Phi} : \mathcal{A}_I \to \mathbb{U}$ *defined by*

$$\mathbb{E}^{\Phi}(A_I) = \mathbb{M}^{\phi}{}_{i=1}^{n}\Pi^{\phi}(A_I(x_i)), \forall x_i \in \chi. \tag{88}$$

Proof. Based on Eqs.(28) and (65), the following holds:

$$\begin{aligned} \mathbb{E}^{\Phi}(A_I(x_i)) &= \mathbb{E}^{\Phi}(A_I) = \phi^{-1}(\mathbb{E}(\phi(l_{\tilde{\mathbb{U}}}(A_I(x_i))), \mathbf{1}-\phi(\mathbf{1}-r_{\tilde{\mathbb{U}}}(A_I(x_i)))) \\ &= \phi^{-1}\mathbb{M}_{i=1}^{n}\Pi(\phi(l_{\tilde{\mathbb{U}}}(A_I(x_i))), \mathbf{1}-\phi(\mathbf{1}-r_{\tilde{\mathbb{U}}}(A_I(x_i)))) \\ &= \phi^{-1}\left(\mathbb{M}_{i=1}^{n}(\phi\circ\phi^{-1})(\Pi(\phi(l_{\tilde{\mathbb{U}}}(A_I(x_i))), \mathbf{1}-\phi(\mathbf{1}-r_{\tilde{\mathbb{U}}}(A_I(x_i)))))\right) \\ &= \phi^{-1}\left(\mathbb{M}_{i=1}^{n}(\phi(\Pi^{\phi}(A_I(x_i)))\right) = \mathbb{M}^{\phi}{}_{i=1}^{n}\Pi^{\phi}(A_I(x_i)) \end{aligned}$$

□

The diagram below summarizes the main results related to the classes of A-GIvIFIx($\mathbb{N}$) and A-IvIFE denoted by $\mathcal{C}(\Pi)$ and $\mathcal{C}(\mathbb{E})$, respectively in figure 9.

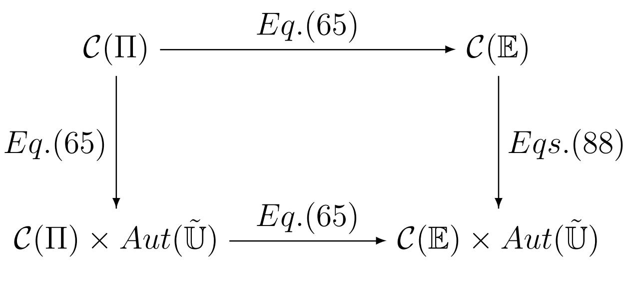

Figure 9 – Conjugate construction of A-GIFIx(N) and A-IvIFE on $Aut(\tilde{U})$

In the following, we extend the above concept of A-IvIFE which is obtained not only from generalized intuitionistic fuzzy index as conceived in (BUSTINCE; BURILLO; SORIA, 2003) but also from their dual and conjugate constructions.

Proposition 24. *Let Φ be a ϕ-representable automorphism in $Aut(\tilde{\mathbb{U}})$ and $\mathbb{I}(\mathbb{J}) : \tilde{\mathbb{U}}^2 \to \tilde{\mathbb{U}}$ be the corresponding ϕ-conjugate operator related to a (co)implication $\mathbb{I}(\mathbb{J}) : \mathbb{U}^2 \to \mathbb{U}$, verifying the conditions of Theorem 5. Additionally, let $\mathbb{N}^\Phi$ be a strong ϕ-conjugate IvFN negation and $\mathbb{M} : \mathbb{U}^n \to \mathbb{U}$ be an aggregation function. Then, for $A \in \mathcal{A}$, the functions $\mathbb{E}_{\mathbb{I}}, \mathbb{E}^\Phi_{\mathbb{I}}(\mathbb{E}_{\mathbb{I}}, \mathbb{E}^\Phi_{\mathbb{I}}) : \mathbb{A} \to \mathbb{U}$ given by*

$$\mathbb{E}_{\mathbb{I}}(A)(x_i) = \mathbb{M}^n_{i=1}\ \mathbb{N}(\mathbb{I}(\mathbf{1} - N_{A_I}(x_i), M_{A_I}(x_i))), \tag{89}$$

$$\mathbb{E}^\Phi_{\mathbb{I}}(A)(x_i) = \mathbb{M}^{\phi}{}^n_{i=1}\, \mathbb{N}^\phi(\mathbb{I}^\phi(\mathbf{1} - N_{A_I}(x_i), M_{A_I}(x_i))); \tag{90}$$

$$\mathbb{E}_{\mathbb{J}}(A)(x_i) = \mathbb{M}^n_{i=1}\ \mathbb{J}(N_{A_I}(x_i), \mathbf{1} - M_{A_I}(x_i)) \tag{91}$$

$$\mathbb{E}^\Phi_{\mathbb{J}}(A)(x_i) = \mathbb{M}^{\phi}{}^n_{i=1}\, \mathbb{J}^\phi(N_{A_I}(x_i), \mathbf{1} - M_{A_I}(x_i)). \tag{92}$$

express an Atanassov's intuitionistic fuzzy entropy .

Proof. Straightforward from Proposition 23. □

Example 12. *By Eqs.(89) and (90), an A-IvIFE expression is obtained as follows:*

$$\mathbb{E}^\Phi_{\mathbb{N}_S,\mathbb{I}_{RB}}(A)(x_i) = \frac{1}{n}\sum_{i=1}^{n}\left[\sqrt[n]{(1-\overline{X}^n_{1i})(1-\overline{X}_{2i})^n};\ \sqrt[n]{(1-\underline{X}^n_{1i})(1-\underline{X}_{2i})^n}\right]. \tag{93}$$

7.0.2 Preserving fuzzyness and intuitionism based on A-IvIFE

Based on (JING; MIN, 2013), assuming that $\chi = \{u\}$, $A_1 = \{(u, [0.1, 0.2], [0.3, 0.4])\}$ and $A_2 = \{(u, [0.2, 0.3], [0.4, 0.5])\}$ in order to calcule the entropies by equations below

$$\mathbb{E}_Y(A) = \frac{1}{n}\sum_{i=1}^{n}\left[\sqrt{2}\, cos\frac{\underline{\mu_A}(x_i) + \overline{\mu_A}(x_i) - \underline{\nu_A}(x_i) - \overline{\nu_A}(x_i)}{8}\pi - 1\right]\frac{1}{\sqrt{2}-1}\ ; \tag{94}$$

$$\mathbb{E}_G(A) = \frac{1}{n}\sum_{i=1}^{n} cos\frac{|\underline{\mu_A}(x_i) - \underline{\nu_A}(x_i)| + |\overline{\mu_A}(x_i) - \overline{\nu_A}(x_i)|}{8}\pi. \tag{95}$$

Thus, $\mathbb{E}(A_1)$ and $\mathbb{E}(A_2)$ contains the difference between the membership and nonmembership degrees related to the hesitancy degree.

However, despite the differences, the same value for related IvIFEs are matched, making it impossible to distinguish the fuzzyness and intuitionism of these two cases.

Intuitively, it is easy to observe that A_1 is more fuzzy than A_2, meaning that $\pi_{A_1} \geq \pi_{A_2}$. However, this cannot be seen by using the above Eqs.(94) and (95). So, a more sensitive definition of A-IvIFE is introduced in order to deal with this problem.

In our proposed methodology, we calculate the related IvIFEs by using Eq.(82) and (93) together with corresponding A-IvIFIx given by Eqs.(75) and (81).

See these results presented in 1st and 2nd columns of Table 15 when the inputs are given as A_1 and A_2. Since χ is singleton A-IvIFS, the resulting hesitant degree and corresponding entropy measure coincide.

Additonally, it is possible to naturally preserve properties of related interval entropy, meaning that A-IvIFE is an order perserving index, by including A-IFE.

Moreover, taking $A_3 = [0.2, 0.2], [0.3, 0.3]$ and $A_4 = [0.3, 0.3], [0.4, 0.4]$ as inputs, the entropy values obtained with the degenerate intervals related to membership and non-membership degrees are included in the interval entropy obtained with non-degenerated interval-valued inputs. See these results in the 3rd and 4th columns of Table 15.

Table 15 – A-IvIFIxs and A-IvIFEs related to A-IvIFSs from A_1 to A_4 A-IvIFSs.

$A - IvIFIx$	A_1	A_2	A_3	A_4
$\Pi(A_i) = \mathbb{E}_{\Pi(A_i)}$	$[0,48; 0,63]$	$[0,35; 0,48]$	$[0,56; 0,56]$	$[0,42; 0,42]$
$\Pi^{\phi}(A_i) = \mathbb{E}^{\phi}_{\Pi}(A_i)$	$[0,5879; 0,6965]$	$[0,4769; 0,5879]$	$[0,4704; 0,4704]$	$[0,3276; 0,3276]$

Through different input sets Entropy, (Eq. 94) and (Eq. 95), obtained equal values, while the Entropy (Eq. 87) obtained through the use of the Reichemback implication showed to be more sensitive with different results, in the same way, we can observe through the results that $\mathbb{E}_{\Pi_{\mathbb{N}_S, \mathbb{I}_{RB}}}$ presented values immediately relevant to the input set for this application.

8 CONCLUSION

This chapter describes the main contribuitions of this work and also points out possible further work.

8.1 Main Contributions

Making use of IFL, this work considers both approaches:

(i) the general concept of the generalized Atanassov's intuitionistic fuzzy index associated with a strong intuitionistic fuzzy negation, which is characterized in terms of fuzzy (co)implication operators as a construction method to model hesitation in intuitionistic fuzzy sets (ZANOTELLI et al., 2016);

(ii) Atanassov's intuitionistic fuzzy entropy proposed in (BUSTINCE et al., 2011) considering aggregation functions applied to the generalized Atanassov's Intuitionistic Fuzzy Index.

Following such results, this work also contributes with extended results in these two related approaches:

(i) introduction of a more general concept of the generalized Atanassov's interval-valued intuitionistic fuzzy index associated with a strong interval-valued intuitionistic fuzzy negation

1. characterized by using interval-valued fuzzy (co)implication operators as a construction method to model hesitation and imprecision in intuitionistic fuzzy sets (COSTA et al., 2017);
2. considered the concept of conjugate and dual interval fuzzy (co)implications, mainly interested in representation method (CORNELIS; DESCHRIJVER; KERRE, 2004) providing relevant properties satisfied by the generated operators.
3. ilustrated results, exploring examples associated with the standard negation together with known fuzzy implications: Lukaziewicz, Reichenbach,

Klenee-Dienes, Gaines-Richard fuzzy (co)implications, preserving main properties and also including operator I_{30} (LIN; XIA, 2006).

(ii) introduction of the concept of interval fuzzy entropy based on the generalized Atanassov's interval-valued intuitionistic fuzzy index and aggregation operators

1. dual and conjugate construction methods are considered in the study of Atanassov's intuitionistic fuzzy entropy (COSTA et al., 2016);
2. examples related to the interval representation of the above (co)implications are discussed, based on the inteval-version of the arthmetic mean agggregator (COSTA et al., 2017);
3. application of admissible linear order providing results in comparison to the methodology for building fuzzy entropy.

In fact, there exist other entropies that we have not considered here. In special, the Szmidt and Kacprzyk's entropy, which relates Atanassov's intuitionistic fuzzy sets and crisp sets.

It is our intention in the future to develop a theoretical framework which allows to consider different approaches to the concept of entropy considering the proposed construction methodology.

Table 16 describes the main publications obtained in this work.

Table 16 – Publications

Publication		**Year**
CNMAC	Generalized Atanassov's Intuitionistic Fuzzy Index and Conjugate with S-implications	2016
ENPOS	Índice Fuzzy Intuicionista Generalizado Conjugado com S-Implicações	2016
CBSF	**Best Paper** - Atanassov's Intuitionistic Fuzzy Entropy: Conjugation and Duality	2016
Mathware & Soft Computing	Analysing Fuzzy Entropy via Generalized Atanassov's Intuitionistic Fuzzy Indexes	2017
WEIT	Truly Intuitionistic Fuzzy Properties of Implications from Generalized Atanassov's Intuitionistic Fuzzy Index	2017
NAFIPS	Interval version of Generalized Atanassov's Intuitionistic Fuzzy Index (submitted)	2018

8.2 Further work

Further work considers the extension of our results related to other properties verified by the generalized Atanassov's interval-valued intuitionistic fuzzy index and the interval extension Atanassov's intuitionistic fuzzy entropy and also the use other admissible linear orders to compare the results of the interval entropy.

We also intend to handle both problems, focusing particularly on how other aggregations can be used to obtain Atanassov's intuitionistic fuzzy entropy based on generalized Atanassov's interval-valued intuitionistic fuzzy index, for instance, Choquet integral allows to define many of the most usual aggregation functions.

Due to the relevance of a theoretical method to calculate the entropy of T2FSs, we leave for a future work the deeper study of an application, i.e., analysing the conditions under which the methodology and illustrative example can improve fuzzy systems in making decision based on multi-attributes.

REFERENCES

ATANASSOV, K. Intuitionistic Fuzzy Sets. **Fuzzy Sets and Systems**, [S.l.], v.20, p.87–96, 1986.

ATANASSOV, K. **Intuitionistic fuzzy sets**: theory and applications. [S.l.]: Physica-Verlag, 1999.

ATANASSOV, K.; GARGOV, G. Interval-valued Intuitionistic Fuzzy Sets. **Fuzzy Sets and Systems**, [S.l.], v.31, p.343–349, 1989.

ATANASSOV, K.; GARGOV, G. Elements of Intuitionistic Fuzzy Logic. **Fuzzy Sets and Systems**, [S.l.], v.9, n.1, p.39–52, 1998.

BACZYNSKI, M. Residual Implications Revisited. Notes on the Smets-Magrez. **Fuzzy Sets and Systems**, [S.l.], v.145, n.2, p.267–277, 2004.

BACZYNSKI, M.; JAYARAM, B. On the characterization of (S,N)-implications. **Fuzzy Sets and Systems**, [S.l.], v.158, n.15, p.1713–1727, 2007.

BARRENECHEA, E. et al. Generalized Atanassov's Intuitionistic Fuzzy Index. Construction Method. **IFSA EUSFLAT Conference**, [S.l.], p.478–482, 2009.

BARRENECHEA, E. et al. Construction of interval-valued fuzzy preference relations from ignorance functions and fuzzy preference relations. Application to decision making. **Knowledge-Based Systems**, [S.l.], v.58, p.33–44, 2014.

BEDREGAL, B. C.; TAKAHASHI, A. Interval Valued Versions of T-Conorms, Fuzzy Negations and Fuzzy Implications. In: IEEE INTERNATIONAL CONFERENCE ON FUZZY SYSTEMS, VANCOUVER, 2006, 2006, Los Alamitos. **Proceedings...** IEEE, 2006. p.1981–1987.

BEDREGAL, B. C.; TAKAHASHI, A. T-normas, T-conormas, Complementos e Implicações Intervalares. **TEMA - Tendências em Matemática Aplicada e Computacional**, [S.l.], v.7, n.1, p.139–148, 2006.

BEDREGAL, B.; SANTIAGO, R.; DIMURO, G.; REISER, R. Interval Valued R-Implications and Automorphisms. In: **Pre-Proceedings of the 2nd Workshop on Logical and Semantic Frameworks, with Applications**. Ouro Preto: UFMG, 2007. p.82–97.

BURILLO, P. J.; BUSTINCE, H. Entropy on intuitionistic fuzzy sets and on interval-valued fuzzy sets. **Fuzzy Sets and Systems**, [S.l.], v.78, n.3, p.305–316, 1996.

BUSTINCE, H.; BARRENECHEA, E.; MOHEDANO, V. Intuicionistic Fuzzy Implication Operators - An Expression and Main Properties. **Uncertainty, Fuzziness and Knowledge-Based Systems**, [S.l.], v.12, p.387–406, 2004.

BUSTINCE, H.; BURILLO, P. J.; SORIA, F. Automorphisms, negations and implication operators. **Fuzzy Sets and Systems**, [S.l.], v.134, n.2, p.209– 229, 2003.

BUSTINCE, H. et al. Generalized Atanassov's Intuitionistic Fuzzy Index: Construction of Atanassov's Fuzzy Entropy from Fuzzy Implication Operators. **International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems**, [S.l.], v.19, n.01, p.51–69, 2011.

BUSTINCE, H. et al. A Historical Account of Types of Fuzzy Sets and Their Relationships. **IEEE Transactions on Fuzzy Systems**, [S.l.], n.1, 2016.

BUSTINCE, H.; FERNANDEZ, J.; KOLESAROVA, A.; MESIAR, R. Generation of Linear Orders for Intervals by Means of Aggregation Functions. **Fuzzy Sets and Systems**, [S.l.], v.220, p.69–77, 2013.

CARLSSON, C.; FULLER, R. **Fuzzy Reasoning in Decision Making and Optimization**. Heidelberg: Physiva-Verlag Springer, 2002.

CHEN, S.-M.; TSAI, W.-H. Multiple attribute decision making based on novel interval-valued intuitionistic fuzzy geometric averaging operators. **Information Sciences**, Taiwan, p.1045–1065, 2016.

CHEN, T.-Y. A Prioritized Aggregation Operator-based Approach to Multiple Criteria Decision Making Using Interval-valued Intuitionistic Fuzzy Sets: A Comparative Perspective. **Information Sciences**, [S.l.], v.281, p.97–112, 2014.

CORNELIS, C.; DESCHRIJVER, G.; KERRE, E. E. Implication in Intuitionistic Fuzzy and Interval-Valued Fuzzy Set Theory: Construction, Classification, Application. **International Journal of Approximate Reasoning**, [S.l.], v.35, n.1, p.55–95, 2004.

COSTA, C. G. da; BEDREGAL, B.; NETO, A. D. D. Relating De Morgan triples with Atanassov's Intuitionistic De Morgan Triples via Automorphisms. **International Journal of Approximate Reasoning**, [S.l.], v.52, p.473–487, 2011.

COSTA, L. et al. Atanassovs Intuitionistic Fuzzy Entropy: Conjugation and Duality. **CBFS 2016, Congresso Brasileiro de Sistemas Fuzzy**, [S.l.], p.1–10, 2016.

COSTA, L. et al. Analysing Fuzzy Entropy via Generalized Atanassov's Intuitionistic Fuzzy Indexes. **Mathware & Soft Computing**, [S.l.], v.42, p.22–31, 2017.

DESCHRIJVER, G.; KERRE, E. E. Implicators based on binary aggregation operators in interval-valued fuzzy set theory. **Fuzzy Sets and Systems**, [S.l.], v.153, n.2, p.229–248, 2005.

DUBOIS, D.; PRADE, H. Random sets and fuzzy interval analysis. **Fuzzy Sets and Systems**, [S.l.], p.87–101, 1991.

DUBOIS, D.; PRADE, H. **Fundamentals of Fuzzy Sets**. Boston: Kluwer Academic Publishers, 2000.

DUGENCI, M. A new distance measure for interval valued intuitionistic fuzzy sets and its application to group decision making problems with incomplete weights information. **Applied Soft Computing**, [S.l.], v.41, p.120 – 134, 2016.

DYMOVA, L.; SEVASTJANOV, P. The Operations on Interval-valued Intuitionistic Fuzzy Values in the Framework of Dempster-Shafer Theory. **Information Sciences**, [S.l.], v.360, n.C, p.256–272, 2016.

FODOR, J.; ROUBENS, M. **Fuzzy Preference Modelling and Multicriteria Decision Support**. Dordrecht: Kluwer Academic Publisher, 1994.

GEHRKE, M.; WALKER, C.; WALKER, E. Some comments on interval valued fuzzy sets. **International Journal of Intelligent Systems**, [S.l.], v.11, n.10, p.751–759, 1996.

GEORGE KLIR, B. Y. Fuzzy Sets and Fuzzy Logics: Theory and Applications. **Prentice Hall PTR**, Upper Saddle River - NJ, 1995.

HUNG, W.; YANG, M. On similarity measures between intuitionistic fuzzy sets. **International Journal of Intelligent Systems**, [S.l.], v.23, p.364–383, 2008.

JING, L.; MIN, S. Some entropy measures of interval-valued intuitionistic fuzzy sets and their applications. **AMO - Advanced Modeling and Optimization**, [S.l.], p.211–221, 2013.

KARNIK, N. N.; MENDEL, J. M. Introduction to type-2 fuzzy logic systems. **Fuzzy Systems Proceedings, IEEE World Congress on Computational Intelligence**, [S.l.], p.915–920, 1998.

KARNIK, N. N.; MENDEL, J. M. Operations on type-2 fuzzy sets. **Fuzzy Sets and Systems**, [S.l.], p.327–348, 2001.

KLEMENT, E. P.; MESIAR, R.; PAP, E. Quasi- and pseudo-inverses of monotone functions, and the construction of t-norms. **Fuzzy Sets and Systems**, [S.l.], v.104, n.1, p.3–13, 1999.

KLEMENT, E. P.; NAVARA, M. A survey on different triangular norm-based fuzzy logics. **Fuzzy Sets and Systems**, [S.l.], v.101, n.2, p.241–251, 1999.

LESKI, J. M. ε-Insensitive Learning Techniques for Approximate Reasoning Systems (Invited Paper). **International Journal of Computational Cognition**, 741 East First Street, Tucson, AZ 85719-4830, USA, v.1, n.1, p.21–77, 2003.

LIN, L.; XIA, Z.-Q. Intuitionistic fuzzy implication operators: Expressions and properties. **Journal of Applied Mathematics and Computing**, [S.l.], v.22, n.3, p.325–338, 2006.

LIU, M.; REN, H. A New Intuitionistic Fuzzy Entropy and Application in Multi-Attribute Decision Making. **Information**, [S.l.], v.5, n.4, p.587–601, 2014.

LUCA, A. de; TERMINI, S. A Definition of a Nonprobabilistic Entropy in the Setting of Fuzzy Sets Theory. **Information and Control**, [S.l.], v.20, n.4, p.301–312, 1972.

MENDEL, J. M. Type-2 Fuzzy Sets: Some Questions and Answers. **IEEE Connections, Newsletter of the IEEE Neural Networks Society**, [S.l.], p.10–13, 2003.

MIGUEL, L. D. et al. Interval-Valued Atanassov Intuitionistic OWA Aggregations Using Admissible Linear Orders and Their Application to Decision Making. **IEEE Transactions on Fuzzy Systems**, [S.l.], v.24, n.6, p.1586–1597, 2016.

MIGUEL, L. D. et al. Type-2 Fuzzy Entropy Sets. **IEEE Transactions on Fuzzy Systems**, [S.l.], v.25, n.4, p.993–1005, 2017.

MOORE, R. E. **Interval Arithmetic and Automatic Error Analysis in Digital Computing**. 1962. Tese (Doutorado em Ciência da Computação) — Stanford University, Stanford.

MOORE, R. E. **Methods and Applications of Interval Analysis**. Philadelphia: SIAM, 1979.

PANKOWSKA, A.; WYGRALAK, M. General IF-sets with triangular norms and their applications to group decision making. **Information Sciences**, [S.l.], v.176, n.18, p.2713–2754, 2006.

QIAN-SHENG; JIANG, Z. S.-Y. A note on information entropy measures for vague sets. **Information Sciences**, [S.l.], v.178, p.4184–4191, 2008.

REISER, R.; BEDREGAL, B.; BACZYNSKI, M. Aggregating fuzzy implications. **Information Sciences**, [S.l.], v.253, p.126–146, 2013.

REISER, R.; DIMURO, G.; BEDREGAL, B.; SANTIAGO, R. Interval valued QL-implications. **WOLLIC. In: D. Leivant, R. Queiroz, (eds.) Language, Information and Computation, Lecture Notes in Computer Science**, Berlin, n.4576, p.307–321, 2007.

REISER, R. H. S.; BEDREGAL, B. C.; REIS, G. A. A. dos. Interval-Valued Fuzzy Coimplications and Related Dual Interval-Valued Conjugate Functions. **Journal of Computer and System Sciences**, [S.l.], 2012.

ROSS, T. J. **Fuzzy Logic with Enginnering Apllications**. England: John Wiley e Sons ltda, 2004.

SAMBUC, R. Function Φ-Flous, Application a laiide au Diagnostic en Pathologie Thyroidienne. **These de Doctorat en Medicine, Univ. Marseille**, France, 1975.

SANCHEZ, M. A.; CASTILLO, O.; CASTRO, J. R. Generalized Type-2 Fuzzy Systems for controlling a mobile robot and a performance comparison with Interval Type-2 and Type-1 Fuzzy Systems. **Expert Systems with Applications**, [S.l.], v.42, n.14, p.5904–5914, 2015.

SILER, W.; BUCKLEY, J. J. **Fuzzy Expert Systems and Fuzzy Reasoning**. New York: John Wiley, 2004.

SZMIDT, E.; KACPRZYK, J. Entropy for intuitionistic fuzzy sets. **Fuzzy Sets and Systems**, [S.l.], v.118, n.3, p.467–477, 2001.

SZMIDT, E.; KACPRZYK, J. A Similarity Measure for Intuitionistic Fuzzy Sets and Its Application in Supporting Medical Diagnostic Reasoning. In: ARTIFICIAL INTELLIGENCE AND SOFT COMPUTING - 7TH INTERNATIONAL CONFERENCE, ZAKOPANE, POLAND, 2004. **Proceedings...** Springer, 2004. p.388–393. (Lecture Notes in Computer Science, v.3070).

TORRA, V. Aggregation operators and models. **Fuzzy Sets and Systems**, [S.l.], v.156, n.3, p.407–410, 2005.

TRILLAS, E.; VALVERDE, L. F. On implication and indistinguishability in the setting of fuzzy logic. In: KACPRZYK, J.; YAGER, R. R. (Ed.). **Management Decision Support Systems using Fuzzy Sets and Possibility Theory**. Cologne: Verlag TUV Rheinland, 1985. p.198–212.

VERMA, R.; SHARMA, B. Exponential entropy on intuitionistic fuzzy sets. **Kybernetika**, Praha, v.49, n.1, p.114–127, 2013.

VLACHOS, I. K.; SERGIADIS, G. D. Intuitionistic fuzzy information - Applications to pattern recognition. **Pattern Recognition Letters**, [S.l.], v.28, p.197–206, 2007.

WAN, S.-P.; WANG, F.; LIN, L.-L.; DONG, J.-Y. Some New Generalized Aggregation Operators for Triangular Intuitionistic Fuzzy Numbers and Application to Multi-attribute Group Decision Making. **Computers & Industrial Engineering**, [S.l.], v.93, n.C, p.286–301, 2016.

WAN, S.-P.; XU, G. li; WANG, F.; DONG, J.-Y. A new method for Atanassov's interval-valued intuitionistic fuzzy MAGDM with incomplete attribute weight information. **Information Sciences**, [S.l.], v.316, p.329–347, 2015.

WANG, X.; ZHU, J.; SONG, Y.; LEI, L. Combination of Unreliable Evidence Sources in Intuitionistic Fuzzy MCDM Framework. **Knowledge-Based Systems**, [S.l.], v.97, n.C, p.24–39, 2016.

WEI, C.; GAO, Z.; GUO, T. ting. An intuitionistic fuzzy entropy measure based on trigonometric function. **Control and Decision**, China, v.27, n.4, p.571–574, 2012.

XU, J.; SHEN, F. A New Outranking Choice Method for Group Decision Making Under Atanassov's Interval-valued Intuitionistic Fuzzy Environment. **Knowledge-Based Systems**, [S.l.], v.70, n.C, p.177–188, 2014.

XU, Z.; YAGER, R. R. Intuitionistic and Interval-Valued Intutionistic Fuzzy Preference Relations and Their Measures of Similarity for the Evaluation of Agreement Within a Group. **Fuzzy Optimization and Decision Making**, [S.l.], v.8, n.2, p.123–139, 2009.

YAGER, R.; KACPRZYK, J. **The Ordered Weighted Averaging Operators**: Theory and Applications. [S.l.]: Springer Publishing Company, Incorporated, 2012.

YAGER, R. R. On Ordered Weighted Averaging Aggregation Operators in Multicriteria Decisionmaking. **IEEE Transactions on Systems, Man, and Cybernetics: Systems**, Piscataway, NJ, USA, v.18, n.1, p.183–190, 1988.

YE, J. Two effective measures of intuitionistic fuzzy entropy. **Computing**, [S.l.], v.87, n.1, p.55–62, 2010.

YUE, Z.; JIA, Y.; YE, G. An approach for multiple attribute group decision making based on intuitionistic fuzzy information. **International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems**, [S.l.], v.17, n.03, p.317–332, 2009.

ZADEH, L. A. Fuzzy Sets. **Information and Control**, University of California, Berkeley, California, v.8, n.3, p.338–353, 1965.

ZADEH, L. A. The Concept of a Linguistic Variable and its Application to Approximate Reasoning - I. **Information Sciences**, University of California, Berkeley, California, v.8, n.3, p.199–249, 1975.

ZANOTELLI, R. et al. Generalized Atanassov's Intuitionistic Fuzzy Index and Conjugate with S-implications. **Proceedings of the 36th Conference of Brazilian Society of Computational and Applied Mathematics, Brazil**, [S.l.], p.1–8, 2016.

ZENG, W.; LI, H. Relationship between similarity measure and entropy of interval valued fuzzy sets. **Fuzzy Sets and Systems**, [S.l.], p.1477–1484, 2006.

ZHANG, H. Entropy for Intuitionistic Fuzzy Sets Based on Distance and Intuitionistic Index. **International Journal of Uncertainty, Fuzziness and Knowledge-Based Systems**, [S.l.], v.21, n.1, p.139–162, 2013.

ZHANG, H.; ZHANG, W.; MEI, C. Entropy of interval-valued fuzzy sets based on distance and its relationship with similarity measure. **Knowledge-Based Systems**, [S.l.], v.22, p.449–454, 2009.

ZHANG, Q.-S.; JIANG, S.-Y. A note on information entropy measures for vague sets and its applications. **Information Sciences**, [S.l.], v.178, n.21, p.4184–4191, 2008.